Supersolubility and some Characterizations of Finite Supersoluble Groups, 2nd Edition

C. J. E. Pinnock

January 1998

Preface

In chapter 1 we introduce the idea of a supersoluble group and we investigate its connexion with other similar concepts such as solubility and nilpotency. In chapter 2 we look at supersoluble series and present some forms ofthese which are common to all supersoluble groups.

The main result of chapter 3 is a conjugacy theorem of Philip Hall regarding Hall π-subgroups in finite groups. In chapter 4 we present some characterization theorems for finite supersoluble groups, including theorems of Huppert, Kramer and Iwasawa. We also give a necessary and sufficient condition for finite supersolubility in terms of the converse of Lagrange's Theorem. Chapter 5 presents some miscellaneous results regarding supersoluble groups.

These notes are essentially my MSci project, which I submitted in March 1997. A few corrections have been made, but some errors remain. The whole project has been reset using LaTeX.

I would like to take this opportunity to thank my supervisor Prof. B. A. F. Wehrfritz for his help and advice on this project, Dr P. H. Kropholler for various discussions and Miss V. M. L. Pryde for her suggestions regarding commas and semicolons.

Chris Pinnock
January 1998

Contents

Introduction

A supersoluble group is a group which can be broken down into cyclic groups by means of a normal series. The class of supersoluble groups sits between the classes of finitely generated nilpotent groups and polycyclic groups. Supersoluble groups are, in some sense, more like nilpotent groups than polycyclic groups. Finite supersoluble groups have some very nice characterizations in terms of their subgroup structure, as we shall see.

We shall need a few preliminary results. In particular, we list some results regarding cyclic groups. As these are the "building blocks" of supersoluble groups, these results ought to be essential in the development of the theory. Results involving automorphism groups of cyclic groups are important because of the normal structure of a supersoluble group.

It is assumed that the reader has a working knowledge of the material in an undergraduate Group Theory course. The contents of [2] is more than ample for our needs. A few well-known results will be referred to by a common name, for example:

- The Modular Law (or Dedekind's Rule) ([11] 7.3).
- Lagrange's Theorem ([12] I.2.j).
- The Isomorphism and Correspondence Theorems ([2] §1).
- Sylow's Theorem ([11] 5.9).
- The Schur-Zassenhaus Theorem ([10] 9.1.2 or [11] 10.30).

Throughout, G will always denote a group. The symbol 1 will be used to denote both the identity of a group and the trivial subgroup, but in a manner that will not cause confusion. We shall write all homomorphisms on the right and shall use the standard notation for subgroups, normal subgroups and presentations. We shall write C_n for the abstract cyclic group of order n, namely $< x : x^n = 1 >$, and $\mathbb{Z}$ for the additive group of integers (the infinite cyclic group up to isomorphism). Sym(n) and Alt(n) denote the symmetric group and alternating group on n letters, respectively. V will be used to denote the group $< (12)(34), (13)(24) >$; that is, the copy of the Klein 4-group inside Alt(4).

By a *series* [1] of G, we mean a finite sequence of subgroups

$$1 = G_0 \leq G_1 \leq ... \leq G_n = G$$

such that $G_i \lhd G_{i+1}$ for all $0 \leq i < n$. The number n is called the *length* of the series, the groups $G_0, G_1, ..., G_n$ are called the *terms* of the series and the quotient groups $G_1/G_0, G_2/G_1, ..., G_n/G_{n-1}$ are called the *factors* of the series. A *normal series* of G is a series whose terms are normal subgroups of G.

A *composition series* of G is a series whose factors are simple. A *chief factor* of G is a quotient H/K where $H, K \lhd G$ and H/K is a minimal normal subgroup of G/K. A *chief series* of G is a normal series whose factors are chief.

Let $\mathcal{P}$ be a property of groups.

A *poly-$\mathcal{P}$ series* is a series whose factors have the property $\mathcal{P}$. G is called *poly-$\mathcal{P}$* if it has a *poly-$\mathcal{P}$ series*. For example, G is called *polycyclic* if it has a series whose factors are cyclic. Note that a group is *soluble* if it is polyabelian.

If $\mathcal{Q}$ is also a property of groups, then G is said to be *$\mathcal{P}$-by-$\mathcal{Q}$* if there is $N \lhd G$ such that N has property $\mathcal{P}$ and G/N has property $\mathcal{Q}$. It is clear that the properties poly-$\mathcal{P}$ and $\mathcal{P}$-by-$\mathcal{Q}$ are preserved by isomorphism provided that the properties $\mathcal{P}$ and $\mathcal{Q}$ are preserved by isomorphism.

If $H \leq G$, we have the *normalizer* of H in G,

$$N_G(H) = \{g \in G : H^g = H\}$$

and the *centralizer* of H in G,

$$C_G(H) = \{g \in G : h^g = h \quad \text{for all} \quad h \in H\}.$$

If $K \lhd H \lhd G$ then G acts by conjugation on H/K in the obvious way. With regard to this action, we have the *centralizer* of H/K in G,

$$C_G(H/K) = \{g \in G : (hK)^g = hK \quad \text{for all} \quad h \in H\}.$$

Note that the *normalizer* in this case is always the whole group G.

We shall denote the group of automorphisms of group X by $\operatorname{Aut} X$.

If $g_1, g_2, ..., g_n \in G$ then we shall write $[g_1, g_2]$ for $g_1^{-1}g_2^{-1}g_1g_2$. Recursively, define

$$[g_1, ..., g_n] = [[g_1, ..., g_{n-1}], g_n],$$

for $n > 2$. If $H_1, H_2, ..., H_n \leq G$, then we shall write $[H_1, H_2]$ for the subgroup

$$< [h_1, h_2] : h_1 \in H_1, h_2 \in H_2 > .$$

Recursively define

$$[H_1, H_2, ..., H_n] = [[H_1, H_2, ..., H_{n-1}], H_n]$$

for $n > 2$. In particular, the *derived subgroup* of G is $G' = [G, G]$.

[1]WARNING: In some literature (e.g. in [13]), what we have called series are called "normal series" and what we shall call normal series are referred to as "invariant series".

$G = \gamma^1 G \geq \gamma^2 G \geq \gamma^3 G \geq ...$ denotes the *lower central series* of G and $1 = \zeta_0 G \leq \zeta_1 G \leq \zeta_2 G \leq ...$ denotes the *upper central series* of G. In particular, $\zeta_1 G$ is the centre of G. In general, a *central series* of G is a series

$$1 = G_0 \leq G_1 \leq ... \leq G_n = G$$

such that $[G_i, G] \leq G_{i-1}$ for all $0 < i \leq n$ or equivalently, a normal series

$$1 = G_0 \leq G_1 \leq ... \leq G_n = G$$

for which $G_i/G_{i-1} \leq \zeta_1(G/G_{i-1})$ for all $0 < i \leq n$. G is called *nilpotent* if it has a central series.

If $H \leq G$, then the *core* of H in G is

$$H_G = \bigcap_{g \in G} H^g$$

which is the largest normal subgroup of G contained in H.

If $H \lhd G$, $K \leq G$, $HK = G$ and $H \cap K = 1$, then we say that K is a *complement* of H in G and that H is a *normal complement* of K in G. Also we say that G is the *semi-direct product* of H by K, denoted $H]K$.

ΦG denotes the *Frattini subgroup* of G, namely the intersection of all maximal subgroups of G, or G if no such subgroups exist. Equivalently, ΦG is the set of all non-generators of G.

$\eta_1 G$ denotes the *Fitting subgroup* of G, namely the subgroup generated by all normal nilpotent subgroups of G.

We list some fairly trivial facts:

0.1 *(a) Let $X, Y \leq G$. Then $(XY : Y)_l = (X : X \cap Y)$.*

(b) If q is the smallest prime dividing the order of finite group G and $H \leq G$ with $(G : H) = q$ then $H \lhd G$.

Proof: (a) An example of a bijection $\{x(X \cap Y) : x \in X\} \longrightarrow \{xY : x \in X\}$ is the map $x(X \cap Y) \longmapsto xY$.

(b) see [10] 1.6.10. □

0.2 *(a) C_n has a unique subgroup of order d, for each divisor d of n. $\mathbb{Z}$ has a unique subgroup of each finite index and these are all the subgroups of $\mathbb{Z}$. Thus, all subgroups of a cyclic group are characteristic.*

(b) Alt(4) *has no subgroup of order* 6 *and V is its only proper non-trivial normal subgroup.* Alt(4) *is polycyclic (i.e. soluble).* □

0.3 The Normalizer/Centralizer Theorem. *Let X be a subgroup or a quotient of a normal subgroup of G. Then there is a homomorphism $N_G(X) \longrightarrow \mathrm{Aut}\, X$, with kernel $C_G(X)$. In particular, $N_G(X)/C_G(X) \hookrightarrow \mathrm{Aut}\, X$.* [2]

[2] $H \hookrightarrow G$ means H can be embedded into G

Proof: The homomorphism is given by $g \mapsto (x \mapsto x^g)$ for $g \in N_G(X)$, $x \in C_G(X)$. □

Theorem 0.4 *Suppose that V is a vector space of dimension $n \geq 1$ over $\mathbb{F}_p$, the field of p elements, and that G is a group of linear automorphisms acting irreducibly on V. If G is abelian of exponent dividing $p-1$ then V has dimension 1.*

Proof: Given $g \in G$, $g^{p-1} = 1$. Thus g satisfies the equation $X^{p-1} - 1 = 0$, which splits over $\mathbb{F}_p$. Thus g has a non-zero eigenvalue $\lambda \in \mathbb{F}_p$. There is a non-zero λ-eigenvector v of g and the λ-eigenspace of g, $W = \{u : ug = \lambda u\}$ is non-trivial. Since G is abelian, $uGg = ugG = \lambda uG$, for every $u \in W$. Thus W is a G-invariant subspace of V. The irreducibility of the G-action gives $W = V$. Hence $ug = \lambda u$ for all $u \in V$ and so the G-action induces scalar multiplication on V. Thus Fv is a G-invariant subspace of V. Therefore $Fv = V$ and so V has dimension 1. □

0.5 *(a) The automorphism group of a cyclic group is a finite abelian group. Furthermore, $|\operatorname{Aut}\mathbb{Z}| = 2$ and for a prime p, $|\operatorname{Aut} C_p| = p-1$.*

(b) Let N be a minimal normal subgroup of finite group G. Suppose N is an elementary abelian p-group. Then $|N| = p$ if and only if $G/C_G(N)$ is abelian of exponent dividing $p-1$.

Proof: (a) See [13], §5.7.

(b) If $|N| = p$ then by 0.3, $G/C_G(N)$ can be embedded into $\operatorname{Aut} N$, which has order $p-1$. Thus $G/C_G(N)$ is abelian of exponent dividing $p-1$. Conversely, let $|N| = p^r$. Since N is an elementary abelian p-group, it may be regarded as the vector space of dimension r over $\mathbb{F}_p$. Since N is a minimal normal subgroup, the group $G/C_G(N)$ regarded as linear transformations of N, acts irreducibly on N. By 0.4, N is cyclic of order p. □

We say that G satisfies *max* if it satisfies the following equivalent conditions:

1) Every non-empty set of subgroups of G has a maximal element.

2) Every subgroup of G is finitely generated.

We say that G satisfies *min* if every non-empty set of subgroups of G has a minimal element.

0.6 *Let $H \leq G$ and $N \lhd G$.*

(a) If G satisfies max *(resp.* min*) then H satisfies* max *(resp.* min*).*

(b) If N and G/N satisfy max *(resp.* min*) then G satisfies* max *(resp.* min*).*

Proof: (a) is clear. For a proof of (b) see [13] 7.1.3. □

0.7 The Schreier Refinement Theorem. *Any two series of G have refinements whose lengths are equal and whose factors are isomorphic in pairs.*

Proof: See [11] 7.7. □

0.8 Fitting's Theorem. *If $M, N \lhd G$ and are nilpotent, then so is MN. It follows that $\eta_1 G$ is nilpotent for finite group G.*

Proof: See [10] 5.2.8. □

0.9 *Let G be a finite group. Then:*

(a) $\Phi G \leq \eta_1 G$.

(b) If $N \lhd G$ *then* $\Phi N \leq \Phi G$.

(c) $\eta_1(G/\Phi G) = \eta_1 G/\Phi G$.

(d) $\eta_1 G = \bigcap \{C_G(H/K) : H/K \quad \text{is a chief factor of} \quad G\}$.

Proof: (a) ΦG is a nilpotent normal subgroup of G.

(b) Suppose that the result is false. Since ΦN is characteristic in N, it is normal in G. There is a maximal subgroup M of G that does not contain ΦN. Thus $G = (\Phi N)M$. And then $N = N \cap G = N \cap (\Phi N)M = (\Phi N)(N \cap M)$. Thus $N \cap M \leq N$. If $N \cap M < N$, let N_1 be a maximal subgroup of N containing $N \cap M$ so that $N = (\Phi N)N_1$. But by definition $\Phi N \leq N_1$, so $N = N_1$, contradiction. If $N \cap M = N$, then $\Phi N \leq N \leq M$, contradiction. Thus the result must be true.

(c) $\eta_1 G$ is nilpotent, so $\eta_1 G/\Phi G$ is nilpotent. It follows that $\eta_1 G/\Phi G \leq \eta_1(G/\Phi G)$. To prove the reverse inclusion, set $N/\Phi G = \eta_1(G/\Phi G)$. Let P be a Sylow p-subgroup of N. $P\Phi G/\Phi G$ is the unique Sylow p-subgroup of $N/\Phi G$, since $N/\Phi G$ is nilpotent. Thus $P\Phi G/\Phi G$ is characteristic in $N/\Phi G$ and so is normal in $G/\Phi G$. Hence $P\Phi G \lhd G$. P is a Sylow p-subgroup of $P\Phi G$.

We claim that $G = N_G(P)\Phi G$. If $g \in G$ then $P^g \leq (P\Phi G)^g = P\Phi G$. So P^g is a Sylow p-subgroup of $P\Phi G$. By Sylow's Theorem, there is $x \in P\Phi G$ with $P^{gx} = P$. Then $gx \in N_G(P)$, and so $g \in N_G(P)x^{-1} \subset N_G(P)P\Phi G = N_G(P)\Phi G$. The reverse inclusion is clear.

Let $\Phi G =< x_1, ..., x_n >$. Then $G = N_G(P)\Phi G =< N_G(P), x_1, ..., x_n >$. The x_i are non-generators of G since they lie in ΦG and so it follows that $G = N_G(P)$. That is, $P \lhd G$. Thus $P \lhd N$. It follows that N is nilpotent and so $N \leq \eta_1 G$. Hence $\eta_1(G/\Phi G) = N/\Phi G \leq \eta_1 G/\Phi G$.

(d) Let $A = \bigcap \{C_G(H/K) : H/K \quad \text{is a chief factor of} \quad G\}$ and choose a chief series of G, say $1 = G_0 < G_1 < ... < G_n = G$. Then

$$1 = G_0 \cap A < G_1 \cap A < ... < G_n \cap A = A$$

is a normal series of A. Further, it is a central series; for $[G_i \cap A, A] \leq A$ and $[G_i \cap A, A] \leq [G_i, A] \leq [G_i, C_G(G_i/G_{i-1})], \leq G_{i-1}$, since we have $C_G(G_i/G_{i-1}) = \{g \in G : [G_i, g] \leq G_{i-1}\}$, so that $[G_i \cap A, A] \leq G_i \cap A$. Thus A is a nilpotent normal subgroup of G, whence $A \leq \eta_1 G$.

Conversely, if H/K is a chief factor of G, it is a minimal normal subgroup of G/K. Now $(\eta_1 G)K/K \lhd G/K$, thus by the minimality of H/K we have either $H/K \cap (\eta_1 G)K/K = 1$ or $H/K \leq (\eta_1 G)K/K$.

In the first case $[H, \eta_1 G] \leq H \cap \eta_1 G \leq (H \cap \eta_1 G)K = H \cap (\eta_1 G)K \leq K$. It follows that $\eta_1 G \leq C_G(H/K)$.

To deal with the second case, note that $(\eta_1 G)K/K \cong \eta_1 G/K \cap \eta_1 G$ is nilpotent. So $H/K \cap \zeta_1((\eta_1 G)K/K) \neq 1$. Thus $H/K \leq \zeta_1((\eta_1 G)K/K)$ so that $[H/K, (\eta_1 G)K/K] = 1$. Hence $[H, \eta_1 G] \leq [H, (\eta_1 G)K] \leq K$ and thus, $\eta_1 G \leq C_G(H/K)$.

It follows that $\eta_1 G \leq A$. $\square$

0.10 *Let G be a finite soluble group. Then:*

(a) A minimal normal subgroup M of G is an elementary abelian normal p-subgroup for some prime p.

(b) Suppose $\Phi G = 1$. Then $\eta_1 G$ is the direct product of (abelian) minimal normal subgroups of G.

(c) $C_G(\eta_1 G) \leq \eta_1 G$.

Proof: (a) M' is normal in G. Since M is soluble, $M' < M$. The minimality of M yields that $M' = 1$. Thus M is abelian. Let p be a prime dividing $|M|$. If M is not a p-group, choose another prime q dividing $|M|$. A Sylow q-subgroup S of M is normal in M, since M is abelian. Thus S is the unique Sylow q-subgroup of M. Hence S is characteristic in M and so $S \lhd G$. This contradicts the minimality of M. Thus M is a p-group. The subgroup $M_p = \{m \in M : m^p = 1\}$, is a non-trivial subgroup of M (M has an element of order p). Also $M_p \lhd G$. Thus $M_p = M$ by the minimality of M. Hence M is an elementary abelian p-group.

(b) Choose L maximal among all subgroups of $\eta_1 G$ which can be expressed as the direct product of minimal normal subgroups of G (note that by (a), $\eta_1 G$ contains all such subgroups). Clearly, $L \lhd G$. Choose $S \leq G$ minimal to the condition that $LS = G$. Since L is abelian, $S \cap L \lhd L$. And $S \cap L \lhd S$, since $L \lhd G$. So $S, L \leq N_G(S \cap L)$ and hence $N_G(S \cap L) \geq SL = G$. That is, $S \cap L \lhd G$.

If $S \cap L \neq 1$ then since $\Phi G = 1$, there is a maximal subgroup M that does not contain $S \cap L$. It follows from the maximality of M that $G = M(S \cap L)$, since $M < M(S \cap L)$. Now $S = S \cap G = S \cap M(S \cap L) = (S \cap M)(S \cap L)$ and $S \cap L \not\subseteq M$. Thus we must have $S \cap M < S$; for otherwise, $S \cap M = S$ implies that $S \cap L = M \cap S \cap L \leq M$, contradiction. Now we have $G = SL = (S \cap M)(S \cap L)L = (S \cap M)L$, but this contradicts the minimality of S to the condition that $SL = G$. Therefore $S \cap L = 1$.

Since $\eta_1 G \lhd G$, we have $S \cap \eta_1 G \lhd S$. Let B be a maximal subgroup of $\eta_1 G$. Since $\eta_1 G$ is nilpotent, $B \lhd \eta_1 G$. Then $\eta_1 G/B$ is a simple nilpotent group and so is abelian. Thus $(\eta_1 G)' \leq B$. Therefore $(\eta_1 G)' \leq \Phi(\eta_1 G) \leq \Phi G = 1$. Thus $\eta_1 G$ is abelian. It follows that $S \cap \eta_1 G \lhd \eta_1 G$. Thus $S \cap \eta_1 G \lhd S\eta_1 G, = G$ since $G = SL \leq S\eta_1 G \leq G$.

If $S \cap \eta_1 G \neq 1$ then there is an abelian minimal normal subgroup H of G such that $H \leq S \cap \eta_1 G$. As $S \cap L = 1$, it follows that $L < L \times H$ (this last product being direct, because $L \cap H \leq L \cap S \cap \eta_1 G$). But this contradicts the maximality of L. Thus $S \cap \eta_1 G = 1$. We now have the result, because $\eta_1 G = SL \cap \eta_1 G = (S \cap \eta_1 G)L = L$.

(c) Set $F = \eta_1 G$ and deny the result. Then $F < FC_G(F)$. Choose a minimal normal subgroup M/F in G/F contained inside $FC_G(F)/F$. Then $M \lhd G$. The solubility of G and minimality of M/F gives $M' \leq F$. Now we have $[\gamma^i F, M] \leq [\gamma^i F, FC_G(F)] = [\gamma^i F, F][\gamma^i F, C_G(F)] \leq (\gamma^{i+1} F)[F, FC_G(F)] = \gamma^{i+1} F$. And F is nilpotent, say $\gamma^{c+1} F = 1$. Then $\gamma^{c+2} M \leq [M', M, ..., M] \leq [F, M, ..., M] = [\gamma^1 F, M, ..., M]$, where M appears c times, $\leq [\gamma^2 F, M, ..., M]$, where M appears $c - 1$ times, $\leq \gamma^{c+1} F = 1$. But then M is a nilpotent normal subgroup and $F < M$, contradiction. □

Chapter 1

Supersolubility

Definition. A *supersoluble series* of G is a normal series of G with cyclic factors. G is called *supersoluble* if it has a supersoluble series.

Trivially, all cyclic groups are supersoluble and all supersoluble groups are polycyclic. However not every polycyclic group is supersoluble; Alt(4) is polycyclic but has no non-trivial normal cyclic subgroups and hence cannot possess a normal series with cyclic factors.

In common with polycyclic, soluble and nilpotent groups, we have:

Proposition 1.1 *Suppose $H \leq G$ and $N \lhd G$, where G is a supersoluble group. Then H and G/N are supersoluble.*

Proof: G has a supersoluble series; that is, a normal series

$$1 = G_0 \leq G_1 \leq ... \leq G_n = G$$

with each G_i/G_{i-1} cyclic. Since each $G_i \lhd G$, each $H \cap G_i \lhd H$ and so we get a normal series of H:

$$1 = H \cap G_0 \leq H \cap G_1 \leq ... \leq H \cap G_n = H.$$

This is a supersoluble series of H because it has cyclic factors; for

$$(H \cap G_i)/(H \cap G_{i-1}) = (H \cap G_i)/((H \cap G_i) \cap G_{i-1})$$

$$\cong (H \cap G_i)G_{i-1}/G_{i-1} \leq G_i/G_{i-1},$$

which is cyclic. Thus H is supersoluble.

Since $N \lhd G$, the subgroups G_iN are normal in G and so by the Correspondence Theorem we have a normal series of G/N,

$$N/N = G_0N/N \leq G_1N/N \leq ... \leq G_nN/N = G/N.$$

This has cyclic factors because using the Isomorphism Theorems

$$(G_iN/N)/(G_{i-1}N/N) \cong G_iN/G_{i-1}N = G_iG_{i-1}N/G_{i-1}N \cong G_i/G_i \cap G_{i-1}N,$$

and

$$G_i/G_i \cap G_{i-1}N \cong (G_i/G_{i-1})/(G_i \cap G_{i-1}N/G_{i-1})$$

which is a quotient of a cyclic group and therefore is cyclic. Hence G/N is supersoluble. □

If $N \lhd G$, and N and G/N are both supersoluble, it is not necessarily true that G is supersoluble; that is to say that an extension of a supersoluble group by a supersoluble group is not always supersoluble. For example, V is a supersoluble normal subgroup of Alt(4) (consider the supersoluble series $1 \leq \ < (12)(34) > \ \leq V$, both of whose factors are isomorphic to C_2) and Alt(4)$/V$ is supersoluble (it is isomorphic to C_3) but as we have already seen, Alt(4) is not supersoluble.

However if $N \lhd G$ and G/N is supersoluble, applying the Correspondence Theorem to a supersoluble series of G/N gives a normal series of G from N up to G with cyclic factors.

If $N \lhd G$ and N has a series whose terms are normal in G and with cyclic factors, then we say that N is *G-supersoluble.*

From the remark and definition we have:

1.2 *If $N \lhd G$, N is G-supersoluble and G/N is supersoluble then G is supersoluble. In particular, a cyclic-by-supersoluble group is supersoluble.* □

A finitely generated abelian group A is supersoluble. Since the trivial group is supersoluble, let $A =< a_1, ..., a_n >$ and inductively assume that abelian groups generated by $n-1$ generators are supersoluble. Now $N =< a_1 > \lhd A$ and $A/N =< a_2N, ..., a_nN >$. By induction, A/N is supersoluble. N is cyclic, so that A is supersoluble by 1.2. We shall deal with a more general situation later - namely that of a finitely generated nilpotent group.

1.3 *Let $N \lhd G$ and G be supersoluble. Then N occurs as a term in a supersoluble series of G.*

Proof: There is a supersoluble series of G/N and hence a supersoluble series between N and G (whose terms are normal in G). Take any supersoluble series of G and intersect it with N to get a G-supersoluble series of N. Put these series together to get the required one. □

Proposition 1.4 *(a) A direct product of finitely many supersoluble groups is supersoluble.*

(b) If $H_1, H_2, ..., H_n$ are normal subgroups of G and the groups $G/H_1, G/H_2, ...,$ C are supersoluble, then $G/\bigcap_{i=1}^{n} H_i$ is supersoluble.

Proof: (a) By using induction, it suffices to show that if G and K are supersoluble then so is $G \times K$. Given supersoluble series

$$1 = G_0 \leq G_1 \leq ... \leq G_n = G$$

and

$$1 = K_0 \leq K_1 \leq ... \leq K_m = K,$$

we note that for all $1 \leq i \leq n$, since $G_i \lhd G$,

$$G_i \times 1 = G_i \times K_0 \lhd G \times K.$$

Similarly, for all $1 \leq j \leq m$,

$$G \times K_j \lhd G \times K.$$

Furthermore, by inspecting the factor groups

$$(G_i \times 1)/(G_{i-1} \times 1) \cong G_i/G_{i-1} \times 1/1 \cong G_i/G_{i-1}$$

for $1 \leq i \leq n$ and

$$(G \times K_j)/(G \times K_{j-1}) \cong G/G \times K_j/K_{j-1} \cong K_j/K_{j-1}$$

for $1 \leq j \leq m$, we see that

$$1 = G_0 \times 1 \leq G_1 \times 1 \leq ... \leq G_n \times 1 = G \times K_0 \leq G \times K_1 \leq ... \leq G \times K_m = G \times K$$

is a supersoluble series of $G \times K$.

(b) Consider the homomorphism $G \longrightarrow \times_{i=1}^{n} G/H_i$, $g \longmapsto (gH_1, gH_2, ..., gH_n)$. It has kernel $\bigcap_{i=1}^{n} H_i$. It follows that $G/\bigcap_{i=1}^{n} H_i \hookrightarrow \times_{i=1}^{n} G/H_i$ which is supersoluble by (a). The result follows by 1.1. □

1.5 *(a) A group is supersoluble if and only if it has a supersoluble series whose factors are infinite or of prime order.*

(b) A supersoluble group has a cyclic normal subgroup of infinite or prime order.

(c) A simple supersoluble group is cyclic of prime order.

Proof: (a) Let G be supersoluble with supersoluble series

$$1 = G_0 < G_1 < G_2 < ... < G_n = G$$

choosing the series to be proper (pick any supersoluble series and throw away repititions of terms). If G_i/G_{i-1} is cyclic of finite but not prime order, we use the following method to refine the series to one that we require. Let p be a prime dividing $|G_i/G_{i-1}|$. Since G_i/G_{i-1} is cyclic, it has a unique cyclic characteristic subgroup H/G_{i-1} of order p. Thus $H/G_{i-1} \lhd G/G_{i-1}$ and so $H \lhd G$. $G_i/H \cong (G_i/G_{i-1})/(H/G_{i-1})$, a quotient of a cyclic group and thus G_i/H is cyclic.

To summarize, we have added a term H between G_{i-1} and G_i in our original supersoluble series with H/G_{i-1} cyclic of prime order and G_i/H cyclic of finite order less than that of G_i/G_{i-1}. We can therefore apply this algorithm repeatedly to get the desired supersoluble series in a finite number of steps.

The converse is trivial.

(b) By (a), every supersoluble group has a supersoluble series whose factors have infinite or prime order. The first non-trivial term in such a series is of the required type.

(c) Let G be supersoluble and simple. By (b), G is cyclic of prime or infinite order. G can't have infinite order because an infinite cyclic group is not simple. □

1.6 *(a) A minimal normal subgroup of a supersoluble group is cyclic of prime order.*

(b) A chief factor of a supersoluble group is cyclic of prime order.

(c) A supersoluble group with a chief series is a finite group.

(d) G is a finite supersoluble group if and only if it has a chief series with cyclic factors of prime order.

Proof: (a) By 1.3, if N is a minimal normal subgroup of G then N is G-supersoluble. But then N must be simple, by minimality. Now apply 1.5 (c).

(b) A chief factor of G, a supersoluble group, is a minimal normal subgroup of some quotient of G. Quotient groups of G are supersoluble by 1.1 and thus by (a), a chief factor of G has prime order.

(c) If supersoluble group G has a chief series then each factor of this series is finite by (b). The order of G is equal to the product of the orders of the factors of this chief series and so G must be finite.

(d) A finite group has a chief series and thus a finite supersoluble group has a chief series with cyclic of prime order factors by (a). Conversely, a chief series with cyclic factors is a normal series with cyclic factors and hence is a supersoluble series. □

Supersoluble groups satisfy the following finiteness condition.

Proposition 1.7 *A supersoluble group satisfies* max.

Proof: A subgroup of a cyclic group is cyclic and in particular finitely generated, so all cyclic groups satisfy *max*. By using induction on the length of a supersoluble series and 0.6(b), a supersoluble group satisfies *max*. □

Since finite groups satisfy *max*, polycyclic-by-finite groups satisfy *max* also.

A supersoluble group does not necessarily satisfy *min*. An easy example is $\mathbb{Z}$; for the set of subgroups $\{2^n\mathbb{Z} : n = 1, 2, 3, ...\}$ has no minimal element.

1.8 *(a) If a supersoluble group G has a composition series then it is finite.*

(b) If a supersoluble group G satisfies min *then it is finite.*

Proof: (a) Given a composition series, each factor is both simple and supersoluble and thus by 1.5(c) is cyclic of prime order. Thus G must be finite.

(b) Since $\mathbb{Z}$ doesn't satisfy *min*, it cannot occur as a factor in any supersoluble series of G by 0.6. Thus any supersoluble series of G has finite factors and so G must be finite.

Alternatively, one can apply (a) by noting that by 1.7, G is a group satisfying both *max* and *min* and therefore has a composition series. □

It follows from 1.7 that maximal subgroups exist in non-trivial supersoluble groups. A subgroup of prime index is clearly maximal. In a supersoluble group, the converse is true.

Theorem 1.9 *The index of a maximal subgroup in a supersoluble group is prime.*

Proof: Let H be a maximal subgroup of G, a supersoluble group. If H is a normal subgroup of G then the result is trivial; for since G/H is supersoluble and simple, by 1.5(c) it must be cyclic of prime order. We can therefore assume that H is not normal in G and put $K = H_G$. Then H/K is a maximal subgroup of supersoluble group G/K and $(G : H) = (G/K : H/K)$. Thus, without loss of generality, we may assume that $K = 1$.

By 1.5(b), the supersolubility of G ensures the existence of a normal subgroup A of G, where A is infinite cyclic or cyclic of prime order. Every subgroup of A is normal in G (by 0.2). Since $H_G = K = 1$, $A \cap H = 1$. So $H < AH$ and by maximality of H we have $G = AH$. If A is infinite then A has a proper non-trivial subgroup B and $H < BH < AH = G$, contradicting the maximality of H. Thus A must be cyclic of prime order. Then

$$(G : H) = (AH : H) = (A : A \cap H) = (A : 1) = |A|,$$

which is prime. □

Later we shall show that if G is a finite group in which each maximal subgroup has prime index, then G is supersoluble.

Theorem 1.10 *Let G be a supersoluble group. Then:*

(a) $\eta_1 G$ is nilpotent and $G/\eta_1 G$ is a finite abelian group.

(b) G is nilpotent-by-(finite abelian). In particular, G' is nilpotent.

Proof: By 1.7, G satisfies *max*. Thus $\eta_1 G$ is finitely generated, say $\eta_1 G =< x_1, x_2, ..., x_n >$. By definition of $\eta_1 G$, each generator x_i lies in a nilpotent normal subgroup, say $x_i \in M_i$, so that

$$\eta_1 G =< x_1, x_2, ..., x_n >\leq M_1 M_2 ... M_n \leq \eta_1 G.$$

So $\eta_1 G = M_1 M_2 ... M_n$, the product of finitely many nilpotent subgroups. By Fitting's Theorem (0.8), $\eta_1 G$ is nilpotent.

Choose a proper supersoluble series of G,

$$1 = G_0 < G_1 < ... < G_n = G.$$

Let $C = \bigcap_{i=1}^{n} C_G(G_i/G_{i-1}), \lhd G$.

Each automorphism group $\mathrm{Aut}(G_i/G_{i-1})$ is finite abelian by 0.5. Also $G/C_G(G_i/G_{i-1}) \hookrightarrow \mathrm{Aut}(G_i/G_{i-1})$ by 0.3. $G/C \hookrightarrow \times_{i=1}^{n} G/C_G(G_i/G_{i-1})$ (cf. proof of 1.4(b)), so that $G/C \hookrightarrow \times_{i=1}^{n} \mathrm{Aut}(G_i/G_{i-1})$. Hence G/C is a finite abelian group. Now each $C_G(G_i/G_{i-1}) = \{g \in G : [G_i, g] \leq G_{i-1}\}$. Thus,

$$[G_i \cap C, C] \leq [G_i, C] \leq [G_i, C_G(G_i/G_{i-1})] \leq G_{i-1}.$$

And

$$[G_i \cap C, C] \leq [C, C] \leq C.$$

Therefore $[G_i \cap C, C] \leq G_{i-1} \cap C$, for $i = 1, ..., n$. Hence the subgroups $G_i \cap C$ give a central series of C, whence C is nilpotent and $C \leq \eta_1 G$.

G/C abelian implies that $G' \leq C$, $\leq \eta_1 G$ and thus G' is nilpotent. Also $G/\eta_1 G$ is abelian. Moreover $(G : \eta_1 G)(\eta_1 G : C) = (G : C)$. Thus $G/\eta_1 G$ is finite. $\square$

Nilpotency is neither a necessary or sufficient condition for supersolubility. For example, $\bigoplus_{\aleph_0} \mathbb{Z}$, the direct sum of a countably infinite number of copies of $\mathbb{Z}$, is a nilpotent group but is not finitely generated, so it cannot be supersoluble by 1.7. Also, Sym(3) is a supersoluble group which is not nilpotent (it has trivial centre). However V is a nilpotent and supersoluble group.

It is natural to search for criteria that ensure that a nilpotent group is supersoluble and it is the condition of a group being finitely generated that distinguishes the supersoluble nilpotent groups from the non-supersoluble nilpotent groups.

Theorem 1.11 *Let G be nilpotent. G is supersoluble if and only if it is finitely generated.*

Proof: By 1.7, supersoluble G is finitely generated. For the converse, suppose $G =< X >$ is a nilpotent group with X a finite set. Set

$$G_n =< [x_1, ..., x_n]^g : \quad \text{each} \quad x_i \in X, g \in G > .$$

We claim that $G_n = \gamma^n G$.

Clearly $G_1 = G = \gamma^1 G$ by definition, so inductively assume that if $n > 1$ then $G_{n-1} = \gamma^{n-1} G$. Every conjugate of every generator of G_n is in G_n so that $G_n \lhd G$. Further $[x_1, x_2, ..., x_n] \in \gamma^n G$ so that $G_{n-1} \leq \gamma^n G$. Set $N = G_n$ and then $H = G/N =< X > /N =< x_i N : x_i \in X >$. Now $[[x_1, ..., x_{n-1}]N, x_n N] = [x_1, ..., x_n]N = N$. Hence every $[x_1, ..., x_{n-1}]N$ centralizes every $x_n N$ in H. That is, every $[x_1, ..., x_{n-1}]N$ centralizes every generator of H and thus it follows that each $[x_1, ..., x_{n-1}]N \in \zeta_1 H$. Then $[x_1, ..., x_{n-1}]^g N \in \zeta_1(H^{gN}) = \zeta_1 H$, so that $\gamma^{n-1} G/G_n = G_{n-1}/N \leq \zeta_1 H = \zeta_1(G/G_n)$. Thus $[\gamma^{n-1} G/G_n, G/G_n] = G_n/G_n$. That is $\gamma^n G = [\gamma^{n-1} G, G] \leq G_n$, completing the proof of the claim.

Clearly $[x_1, ..., x_n]^g = [x_1, ..., x_n][x_1, ..., x_n, g]$ and $[x_1, ..., x_n, g] \in G_{n+1} = \gamma^{n+1} G$, thus we see that $\gamma^n G/\gamma^{n+1} G$ is generated by all elements

$$[x_1, ..., x_n]^g \gamma^{n+1} G = [x_1, ..., x_n] \gamma^{n+1} G$$

. Since X is finite, $\gamma^n G/\gamma^{n+1}G$ is generated by the finite set $\{[x_1, ..., x_n]\gamma^{n+1}G : x_i \in X\}$. Suppose that $\gamma^n G/\gamma^{n+1}G =< y_i\gamma^{n+1}G : i = 1, ..., r >$. Set $K_i =< \gamma^{n+1}G, y_1, ..., y_i >$. Then for each i, since $y_1, ..., y_i \in \gamma^n G$ and $\gamma^{n+1}G \leq \gamma^n G$ we have $[K_i, G] \leq [\gamma^n G, G] = \gamma^{n+1}G$. Hence $K_i/\gamma^{n+1}G \lhd G/\gamma^{n+1}G$ and $K_i \lhd G$. Further $K_i/K_{i-1} =< y_i K_{i-1} >$ which is cyclic. Thus we have constructed a series with cyclic factors whose terms are normal in G from $\gamma^{n+1}G$ to $\gamma^n G$, viz.

$$\gamma^{n+1}G = K_0 \leq K_1 \leq ... \leq K_r = \gamma^n G$$

for any n. Since G is nilpotent, $\gamma^d G = 1$ for some integer d. Thus we have found a supersoluble series of G. Hence G is supersoluble. □

1.11 together with 1.7 gives:

Corollary 1.12 *Every finitely generated nilpotent group satisfies* max. □

We can summarize some of the results of this chapter by means of a diagram:

f.g. nilpotent	⇒	**supersoluble**	⇒	polycyclic ≡ soluble + *max*
⇓				⇓
nilpotent		⇒		soluble

To see that every soluble group G satisfying *max* is polycyclic, note that the factors of any soluble series of G must be finitely generated, are therefore finitely generated abelian groups and thus they are finite direct products of cyclic groups. We can therefore refine a soluble series of G to a polycyclic series.

The converses of the above implications are not true. For example, we have seen previously that Alt(4) is a polycyclic group which is not supersoluble and that Sym(3) is a supersoluble group that is not nilpotent.

1.10 says that a supersoluble group is nilpotent by finite abelian. Therefore the notion of a supersoluble group is nearer to that of a finitely generated nilpotent group than to that of a polycyclic group.

If we consider only finite groups, the above diagram reduces to the following:

nilpotent ⇒ **supersoluble** ⇒ soluble ≡ polycyclic

Chapter 2

Supersoluble Series

The main goal of this section is to specify certain forms of supersoluble series that are common to all supersoluble groups. The strategy here is to take a supersoluble series of a group, "rearrange" its factors and produce another supersoluble series which has a nicer form.

Given a supersoluble series

$$1 = G_0 \leq G_1 \leq ... \leq G_n = G,$$

to avoid complication, we shall say that the "factors from left to right" are G_1/G_0, G_2/G_1, ..., G_n/G_{n-1} As we shall be referring to the order of the factors in this way, we shall avoid confusion by sometimes calling the order of a group its *magnitude.*

Every supersoluble group has a useful numerical invariant.

Theorem 2.1 *Any two supersoluble series of group G have the same number of infinite factors.*

In fact, the same result holds for any two polycyclic-by-finite series of a polycyclic-by-finite group. We call this invariant the *Hirsch number* [1] of G.

Proof: By the Schreier Refinement Theorem (0.7), any two supersoluble series of supersoluble group G have refinements whose factors are isomorphic in pairs. We can therefore complete the proof by showing that a supersoluble series of G and any of its refinements have the same number of infinite factors.

Suppose

$$1 = G_0 \leq G_1 \leq ... \leq G_n = G$$

is a supersoluble series with G_i/G_{i-1} infinite cyclic for some i. Suppose further that

$$G_{i-1} = H_0 < H_1 < ... < H_m = G_i$$

[1] after Kurt August Hirsch (1906-1986), first Professor of Pure Mathematics at Queen Mary College, University of London. He published several papers on infinite soluble groups and was the first to seriously study polycyclic groups.

with each $H_j \lhd G$ and each H_j/H_{j-1} non-trivial. Now $1 < H_1/H_0 \leq G_i/H_0 = G_i/G_{i-1} \cong \mathbb{Z}$ so that H_1/H_0 is infinite cyclic. Moreover it is isomorphic to a non-trivial subgroup of $\mathbb{Z}$ and thus has finite index in G_i/G_{i-1}. Since

$$(G_i/G_{i-1} : H_1/H_0) = (H_m/H_0 : H_1/H_0) = |(H_m/H_0)/(H_1/H_0)| = |H_m/H_1|,$$

H_m/H_1 is a finite group. We have shown that a supersoluble series and its refinements have the same number of infinite cyclic factors and so the result follows. □

By 1.5 every supersoluble group has a supersoluble series with its factors infinite or of prime order. We now look at ways of "arranging" such factors in a supersoluble series.

2.2 The First Rearrangement Lemma. *Let $1 < H < K < G$ be a normal series of G with H and K/H cyclic.*

(a) If $|H| = q < p = |K/H|$, where p and q are primes, then there is $R \lhd G$ with $R \leq K$ such that $|R| = p$ and $|K/R| = q$.

(b) If H is infinite and K/H has odd prime order p then either K is infinite cyclic or there is $R \lhd G$ with $R \leq K$ such that $|R| = p$ and K/R is infinite cyclic.

(c) If H has order 2 and K/H is infinite then there are $R_1, R_2 \lhd G$ with $R_1 < R_2 < G$, R_1 infinite cyclic and both $R_2/R_1, K/R_1$ are cyclic of order 2.

Proof: (a) K must have order pq. Let R be the Sylow p-subgroup of K (uniqueness is given by the fact that the number of Sylow p-subgroups of K is congruent to 1 modulo p, divides the prime q and $q < p$). R is normal in G. Furthermore $|R| = p$ and $|K/R| = pq/p = q$.

(b) H is an abelian group and therefore $H \leq C_K(H)$. Since H is normal in K, there is a homomorphism $\phi : K \longrightarrow \operatorname{Aut} H$ with kernel $C_K(H)$, by 0.3. Since $H \leq Ker\phi$, the map $\phi' : K/H \longrightarrow \operatorname{Aut} H$, $kH \longmapsto k\phi$ for $k \in K$, is a well-defined homomorphism whose kernel is $C_K(H)/H$. But $K/H \cong C_p$, p an odd prime and $\operatorname{Aut} H \cong C_2$, so that ϕ' must be trivial. Thus $C_K(H)/H = K/H$ and so $K = C_K(H)$. Hence $H \leq \zeta_1 K \leq K$.

The simplicity of $K/H \cong C_p$ implies that $\zeta_1 K = K$ or H. $K/\zeta_1 K$ is never a non-trivial cyclic group (for any group K), so $\zeta_1 K \neq H$. Thus K is abelian. We note also that K is supersoluble of Hirsch number 1. By 1.7, K is a finitely generated abelian group. Let T be its torsion subgroup. T is characteristic in K and thus is a normal subgroup of G. K/T is a direct product of a finite number of copies of $\mathbb{Z}$; for K/T is a torsion-free finitely generated abelian group. Also K/T is supersoluble and must have Hirsch number 1, since T is finite. Thus $K/T \cong \mathbb{Z}$. Since H is torsion-free, $T \cap H = 1$. Thus

$$T \cong T/T \cap H \cong TH/H \leq K/H \cong C_p.$$

Then $T = 1$ or $T \cong C_p$. If $T = 1$, that is if K is torsion-free, then $K \cong K/T \cong \mathbb{Z}$. If otherwise, put $R = T$; for then $R \lhd G$, $K/R \cong \mathbb{Z}$ and $R \cong C_p$.

(c) K/H is infinite cyclic and so is generated by some Hk where $k \in K$ and k has infinite order. Thus $K = H < k >$. Since H is finite and $< k >$ is infinite, $H \cap < k >= 1$. Also since H is cyclic of order 2, $H =< h >$, where $h^2 = 1$. Since h^k must have order 2 and also lie in H, we have $h^k = h$, so that $[H, k] = 1$. It follows that $K = H \times < k >$. Set $R_1 = K^2 =< x^2 : x \in K >$. Since K is a normal subgroup of G and $(x^2)^g = (x^g)^2$ for every $g \in G$, R_1 is a normal subgroup of G. Moreover, $R_1 = (H \times < k >)^2 =< k^2 >$ since H has order 2. Thus R_1 is infinite cyclic. Note that

$$|K/R1| = |(H \times < k >)/ < k^2 > | = |H \times C_2| = 4.$$

Set $R_2 = HK^2$. Then $R_1 < R_2 < K$. Since H and K^2 are normal subgroups of G, R_2 is a normal subgroup of G. Further,

$$|R_2/R_1| = |HK^2/K^2| = |H/H \cap K^2| = |H| = 2,$$

as $H \cap K^2 = 1$, and

$$|K/R_2| = |(K/R_1)/(R_2/R_1)| = 4/2 = 2.$$

□

Let

$$1 = G_0 < G_1 < ... < G_n = G$$

be a supersoluble series of G. For $0 < i < n$ we have a normal series

$$1 = G_{i-1}/G_{i-1} < G_i/G_{i-1} < G_{i+1}/G_{i-1} < G/G_{i-1}$$

on which we can apply 2.2. Informally the result says that given neighbouring factors in a supersoluble series, to produce a new supersoluble series we can

(a) shift a factor of prime order q to the right of a factor of prime order p provided that $p > q$;

(b) shift a factor of infinite order to the right of a factor of odd prime order p possibly at the expense of losing the factor of order p;

(c) shift a factor of order 2 to the right of an infinite factor at the expense of inserting another factor of order 2 to the right of the infinite factor.

We are now in a situation to give our first canonical form.

Theorem 2.3 *(*Zappa) A supersoluble group G has a supersoluble series in which the cyclic factors have infinite or prime order and the order of the factors from the left is:

- factors of odd magnitude in descending order of magnitude;

- infinite factors;
- factors of order 2.

Proof: Using 1.5(a), G has a supersoluble series whose factors have infinite or prime order. By the previous discussion we can use 2.2 to get the required supersoluble series. Using 2.2(a) and 2.2(c) we can produce a supersoluble series whose factors of order 2 are last. Then by 2.2(b) we can get a supersoluble series whose factors of order 2 are last, preceded by its infinite factors. Finally, one can use 2.2(a) to order the factors of odd prime order. □

Corollary 2.4 *The elements of odd order in a supersoluble group form a characteristic subgroup.*

Proof: Choose a supersoluble series of G as in 2.3, say

$$1 = G_0 < G_1 < ... < G_r < G_{r+1} < ... < G$$

where G_{r+1}/G_r is the first infinite factor. Then clearly G_r is a subgroup of G consisting precisely of the elements of odd order in G. Automorphisms take elements of odd order to elements of odd order. The result follows. □

Since a finite supersoluble group has Hirsch number 0, we have:

Corollary 2.5 *If G is a finite supersoluble group then G has a supersoluble series*

$$1 = G_0 < G_1 < ... < G_n = G$$

with each $|G_i/G_{i-1}|$ prime and $|G_1/G_0| \geq |G_2/G_1| \geq ... \geq |G_n/G_{n-1}|$. □

We now give some examples to illustrate why 2.2 is, in some sense, the best possible result we can hope for.

(i) We cannot necessarily produce a supersoluble series of a group in which the finite factors are in ascending order of magnitude. This is because we cannot always shift a factor of prime order q to the right of a factor of prime order p when $q > p$. To see this, note that Sym(3) has a unique supersoluble series $1 \leq < (123) > \leq$ Sym(3) with factors from left to right C_3, C_2 so that there is not a supersoluble series of Sym(3) whose factors are in ascending order of magnitude.

(ii) We cannot necessarily move a factor of order 2 to the left of an infinite factor. An example can be found by looking at the infinite dihedral group $D_\infty =< x, y : x^2 = 1, y^x = y^{-1} >$, of the form $\mathbb{Z}]C_2$. One can show that D_∞ has no normal subgroups of order 2 and thus has no supersoluble series whose factors from left to right are C_2 then $\mathbb{Z}$.

(iii) One cannot necessarily shift a factor of odd prime order p to the right of an infinite factor without introducing another finite factor of order not p. For example, consider the group G with presentation $< x, y : x^3 =$

$1, x^y = x^{-1} >$. It is easy to show that this group is a semi-direct product $< x >] < y >$ - i.e. of the form $C_3]\mathbb{Z}$. Furthermore, the infinite elements of G have one of the forms y^i, xy^i, x^2y^i (where $i \in \mathbb{Z}$). It is fairly routine to show that for z an element of infinite order in G, $< z > \lhd G$ if and only if $z = y^{2i}$ for i an integer. It follows that the normal infinite cyclic subgroup of G of smallest index is $< y^2 >$ which has index 6. It follows also that a supersoluble series of G with an infinite factor first must have some factor isomorphic to C_2.

We do have a method of "moving" infinite factors to the left of a finite factor by means of a more general result.

2.6 The Second Rearrangement Lemma. *If $1 < H < K < G$ is a normal series of G with H finite and K/H infinite cyclic, then there is a normal subgroup R of G contained in K such that R is infinite cyclic and K/R is finite.*

Proof: H is a normal subgroup of K, so that $K/C_K(H) = N_K(H)/C_K(H)$ can be embedded into Aut H. Since H is a finite group, Aut H is finite. Thus $K/C_K(H)$ is finite. And $1 < \zeta_1 H \leq C_K(H) \leq K$. So we may as well assume that K centralizes H so that $1 < H \leq \zeta_1 K \leq K$. Since K/H is cyclic, $K/\zeta_1 K$ is cyclic and thus K is abelian. So consider the normal series $1 < H \leq K \leq G$ with K abelian.

We have K/H generated by some element Hx where $x \in K$ has infinite order. Thus $K = H < x >$. K is abelian, so $[H, x] = 1$ and H is finite so $H \cap < x >= 1$. Thus $K = H \times < x >$. Let $n = |H|$. Take $R = K^n$. Since $H^n = 1$, we have $R =< x^n >$. Then R is infinite cyclic. Since $(x^g)^n = (x^n)^g$ for every $g \in G$, it follows that R is a normal subgroup of G. Finally,

$$|K/R| = |H < x > / < x^n > | = |H|n = n^2.$$

Thus $|K/R|$ is finite as required. □

Theorem 2.7 *If G is a supersoluble group then it has a supersoluble series with the infinite factors appearing first.*

Proof: By 2.1, we can induct on the Hirsch number m of a supersoluble group G. If $m = 0$ then any supersoluble series of G satisfies the required property. Suppose that $m > 0$ and that for supersoluble groups of Hirsch number $m - 1$ the result holds. Let

$$1 = G_0 < G_1 < ... < G_n = G$$

be a proper supersoluble series of G. Choose r to be the smallest integer such that G_r/G_{r-1} is infinite cyclic. Clearly $r > 0$.

If $r = 1$ then G/G_1 is a supersoluble group with Hirsch number $m - 1$. By induction, there is a supersoluble series

$$G_1/G_1 = H_1/G_1 < H_2/G_1 < ... < H_s/G_1 = G/G_1$$

with the infinite factors appearing first. Then

$$1 = G_0 < G_1 = H_1 < H_2 < ... < H_s = G$$

is a supersoluble series of G with the infinite factors appearing first.

If $r > 1$ apply 2.6 to the normal series

$$1 < G_{r-1} < G_r < G$$

to obtain a normal subgroup R of G contained in G_r such that R is infinite cyclic and G_r/R is finite. G/R is therefore a supersoluble group with Hirsch number $m - 1$, so by induction there is a supersoluble series of G/R and thus there is a normal series of G with cyclic factors between R and G with the infinite factors first. This series, together with 1 and R, gives a supersoluble series of G with the infinite factors first. □

Corollary 2.8 *(a) A supersoluble group has a normal poly-(infinite cyclic) subgroup of finite index.*

(b) An infinite supersoluble group has a non-trivial torsion-free abelian normal subgroup.

Proof: (a) follows directly from 2.7. To show (b), let L be a normal poly-(infinite cyclic) subgroup of supersoluble G, as in (a). Then L is certainly soluble. Let T be the last non-trivial term in the derived series of L. T is an abelian group, it is torsion-free and is finitely generated. It is characteristic in L and so is normal in G. □

More generally, polycyclic-by-finite groups always have a polycyclic-by-finite series in which the infinite cyclic factors appear first and therefore has a poly-(infinite cyclic) subgroup of finite index. The proof of 2.8(b) also generalizes so that an infinite polycyclic-by-finite group have a non-trivial torsion-free abelian normal subgroup.

Corollary 2.9 *If G is a supersoluble group then G has a supersoluble series in which each factor is infinite cyclic or cyclic of prime order, and such that the infinite factors appear first, then the finite factors in descending order of magnitude.*

Proof: By 2.7, G has a supersoluble series

$$1 = G_0 < G_1 < ... < G_n = G$$

with the infinite factors first. Let r be the largest integer such that G_r/G_{r-1} is infinite cyclic. Then G/G_r is a finite group. By 2.5, there is a supersoluble series

$$G_r/G_r = H_{r+1}/G_r < H_{r+2}/G_r < ... < H_{r+s}/G_r = G/G_r$$

with the factors of prime order and in descending order of magnitude. Then:

$$1 = G_0 < G_1 < ... < G_r = H_{r+1} < H_{r+2} < ... < H_{r+s} = G$$

is a supersoluble series of G. The condition on the finite factors holds because

$$|H_{r+i}/H_{r+i-1}| = |(H_{r+i}/G_r)/(H_{r+i-1}/G_r)|$$

for $i = 1, 2, ..., s$. □

As we have seen, some of the results of this section hold more generally for polycyclic-by-finite groups and polycyclic-by-finite series. We end this section by showing that 2.3, 2.4, 2.5 and 2.9 do not generalize. For counterexamples we rely on our standard example of a polycyclic group which is not supersoluble, Alt(4).

Alt(4) has only one proper non-trivial normal subgroup V and $\mathrm{Alt}(4)/V \cong C_3$. V has three elements of order 2 and so it follows that Alt(4) has only 3 polycyclic series, all of whose factors from left to right are (up to isomorphism) C_2, C_2 and C_3. We list these:

$$1 \leq < (12)(34) > \leq V \leq \mathrm{Alt}(4)$$

$$1 \leq < (13)(24) > \leq V \leq \mathrm{Alt}(4)$$

$$1 \leq < (14)(23) > \leq V \leq \mathrm{Alt}(4)$$

Clearly Alt(4) has no polycyclic series with the factors in descending order of magnitude. Also the elements of odd order in Alt(4) don't form a subgroup; for example, $(123)(234) = (13)(24)$ which has even order. This kills any hope of the aforementioned results being true for polycyclic groups and thus any hope of them being true for polycyclic-by-finite groups.

Chapter 3

Sylow Towers and a Theorem of Philip Hall

Throughout this chapter, all groups will be **finite**.

Definition. Let $p_1, ..., p_r$ be the distinct prime divisors of $|G|$. A *Sylow tower of complexion* $(p_1, ..., p_r)$ of G is a sequence of subgroups of $G_1, ..., G_r$ of G such that G_i is a Sylow p_i-subgroup of G for each $i = 1, ..., r$ and $G_1 G_2 ... G_k \lhd G$ for each $k = 1, ..., r$. Note that given such subgroups $G_1, ..., G_r$, $G_1 G_2 ... G_r$ has the order of G and thus must be G itself.

If the prime divisors are ordered so that $p_1 > p_2 > ... > p_r$, then we shall call a Sylow tower of complexion $(p_1, ..., p_r)$ of G just a *Sylow tower* of G.

Proposition 3.1 *Every supersoluble group has a Sylow tower.*

Proof: Let G be supersoluble. We induct on the number of prime divisors of $|G|$. If G is trivial then the result clearly holds. So assume that G is non-trivial.

Let $p = p_1 > p_2 > ... > p_r$ be the distinct prime divisors of $|G|$. Clearly, a supersoluble series of G whose factors have prime order must include some factor of order p. By 2.5, there is a supersoluble series of G in which the factors of order p appear first, say

$$1 = G_0 < G_1 < ... < G_n = G.$$

If r is chosen maximal to the condition that $|G_r/G_{r-1}| = p$ then G_r is a normal subgroup of G of order p_r. Furthermore, any prime dividing $(G : G_r)$ is strictly less than p. Thus $S = G_r$ is a normal Sylow p-subgroup of G.

By induction, G/S has a Sylow tower (of complexion $(p_2, ..., p_r)$), say

$$T_2/S, ..., T_r/S.$$

Note that $T_2, T_2T_3, ..., T_2T_3...T_r$ are all normal subgroups of G. For i = 2, ..., r, let S_i be a Sylow p_i-subgroup of T_i and $S_1 = S$. Since $|T_i| = |S|p_i^e$, where p_i^e

is the power of p_i dividing $|G|$, it follows that each S_i is a Sylow p_i-subgroup of G. And further $S_1 \lhd G$ (trivially), $S_1S_2 = T_2 \lhd G$, ..., $S_1S_2...S_r = T_2...T_r \lhd G$. Thus G has a Sylow tower. □

Corollary 3.2 *(a) If G is a supersoluble group and p is the largest prime dividing $|G|$ then G has a normal Sylow p-subgroup S. Further, S has a complement in G.*

(b) If G is a supersoluble group and p is the smallest prime dividing $|G|$ then a Sylow p-subgroup P of G has a normal complement in G.

Proof: By 3.1, G has a Sylow tower, say $G_1, G_2, ..., G_r$. To prove (a), take $S = G_1 \lhd G$. For then, since $(G : S)$ and $|S|$ are coprime, the Schur-Zassenhaus Theorem says that S has a complement in G. To prove (b), take $P = G_r$ and $Q = G_1...G_{r-1} \lhd G$. Since the orders of $G_1, ..., G_{r-1}$ and the order of G_r are coprime, it follows from Lagrange's Theorem that $Q \cap P = 1$. Clearly $QP = G$, so Q is a complement of P in G. Q is normal in G so that it is a normal complement of P in G. □

If G has a Sylow tower then it is not necessarily supersoluble. For example, Alt(4) has Sylow tower $V, <(123)>$. There is a property which in addition to a group having a Sylow tower, characterizes (finite) supersoluble groups. We state the relevant Theorem but do not prove it (see [16] Theorem 1.12, page 6)

Definition. Let p be a prime. A group K is called *strictly p-closed* if K has a unique (and thus normal) Sylow p-subgroup T and K/T is abelian of exponent dividing $p-1$. We shall see later that a strictly p-closed group, for some prime p, is supersoluble.

Theorem 3.3 (Baer). *G is supersoluble if and only if*

(a) G has a Sylow tower.

(b) Given any prime p and any Sylow p-subgroup S of G, $N_G(S)/C_G(S)$ is strictly p-closed.

Definition. Let π be a set of prime numbers. Let π' denote the set of all primes that do not occur in π. A *π-number* is a number divisible only by primes in π. A *π-group* (resp. *π-subgroup*) is a group (resp. subgroup) whose order is a π-number. A *Hall π-subgroup* of G is a π-subgroup H of G such that $(G : H)$ is a π'-number. Note that if $\pi = \{p\}$, p a prime then a π-group is precisely a p-group and a Hall π-subgroup is precisely a Sylow p-subgroup; so these notions generalize the notion of a Sylow p-subgroup.

Sylow's Theorem establishes the conjugacy (and hence isomorphism) of the Sylow p-subgroups of a group G. Much research has been done into the conjugacy of other "special" subgroups, such as Hall π-subgroups. The main theorem of this section is a result regarding these.

Theorem 3.4 (P. Hall [5]) *Let G be a group. Any two supersoluble Hall π-subgroups of G are conjugate in G.*

This follows from a more general result:

Theorem 3.5 (P. Hall [5]) *Let G be a group and π be a set of primes. Let $p_1, ..., p_r$ be the distinct primes in π that divide $|G|$. Let H, K be Hall p-subgroups of G both with Sylow towers of complexion $(p_1, ..., p_r)$. Then H and K are conjugate in G.*

Proof: Let $S_1, ..., S_r$ and $T_1, ..., T_r$ be Sylow towers of complexion $(p_1, ..., p_r)$ for H and K respectively. We induct on r. If $r = 1$ then H and K are Sylow p_1-subgroups of G and are conjugate by Sylow's Theorem. Assume that $r > 1$ and put $H_1 = S_1S_2...S_{r-1}$ and $K_1 = T_1T_2...T_{r-1}$. By definition of Sylow tower, H_1 is a normal subgroup of H and K_1 is a normal subgroup of K. Also H_1 and K_1 have Sylow towers of complexion $(p_1, ..., p_{r-1})$ and are Hall $\{p_1, ..., p_{r-1}\}$-subgroups of G. Hence by induction H_1 and K_1 are conjugate. Thus without loss of generality we may assume that $H_1 = K_1$, replacing K and the T_i's by conjugates if necessary (anything conjugate to this "new" K will be conjugate to the "old" K since conjugacy is a transitive relation). Let p_r^e be the highest power of p_r dividing G. Since the subgroups $S_1, ..., S_{r-1}$ have orders which do not involve the prime p_r, $S_1S_2...S_{r-1} \cap S_r = 1$ using Lagrange's Theorem. Then

$$|H/H_1| = |S_1S_2...S_r/S_1...S_{r-1}| = |S_r|$$

By using an Isomorphism Theorem, $|S_r| = p_r^e$. In a similar way, $|K/H_1| = p_r^e$. H_1 is normal in both H and K so that H and K are contained in $N_G(H_1)$. It follows that H/H_1 and K/H_1 are Sylow p_r-subgroups of $N_G(H_1)/H_1$. By Sylow's Theorem there is $g \in N_G(H_1)$ such that $H^g/H_1 = K/H_1$ and thus $H^g = K$, as required. □

Proof of 3.4: Let $p_1, ..., p_r$ be the distinct primes in π that divide $|G|$ and choose them so that $p_1 > p_2 > ... > p_r$. The distinct primes dividing $|H|$ and $|K|$ are amongst $p_1, ..., p_r$. Thus, 3.1 yields Sylow towers of complexion $(p_1, ..., p_r)$ for both H and K. By 3.5, H and K are conjugate. (One should note that if p, a prime, does not divide $|J|$, for a group J , then a Sylow p-subgroup of J is trivial.) □

Chapter 4

Some Characterization Theorems for finite Supersoluble groups

Throughout this chapter, all groups will be **finite**.

If G is an abelian group then for any divisor n of $|G|$ there is a subgroup H of G for which $|H| = n$. Of course, in this case any subgroup of G is normal.

It is fairly straightforward to show that a group G is nilpotent if and only if for every divisor n of $|G|$ there is a normal subgroup N of G with $|N| = n$. This is the content of a short paper by C. V. Holmes ([6]).

We now present a similar characterization for supersoluble groups, giving a similar proof to one of W. E. Deskins' ([3]).

Definition. We shall say that G satisfies (or is) *clt* if it satisfies the converse of Lagrange's Theorem. That is to say that G satisfies *clt* if whenever n divides $|G|$, G has a subgroup of order n.

Alt(4) does not satisfy *clt* because it has no subgroup of order 6. Sym(4) does satisfy *clt*:

Order	1	2	3	4	6	8	12	24
Subgroup	1	C_2	C_3	V	Sym(3)	D_8	Alt(4)	Sym(4)
		(up to isomorphism)						

Sym(4) contains Alt(4), and so we note that a subgroup of a *clt* group is not necessarily *clt*.

One can show that every *clt* group is necessarily soluble (see for example [16] Theorem 1.4, page 71). We will show that every supersoluble group is *clt*. Since any subgroup of a supersoluble group G is supersoluble, every subgroup of G must be *clt*. It turns out that this last property is a sufficient condition for supersolubility.

Lemma 4.1 *The following conditions are equivalent.*

(a) Every subgroup of G satisfies clt.

(b) If $H \leq G$ then for every prime divisor p of $|H|$, there is a subgroup $K \leq H$ with $(H : K) = p$.

Proof: $(a) \Rightarrow (b)$ If p is a prime dividing $|H|$ then $|H|/p$ is an integer dividing $|H|$. By hypothesis, there is a subgroup K of H whose order is $|H|/p$ and hence has index p in H.

$(b) \Rightarrow (a)$ Suppose n divides $|H|$, where $H \leq G$. Then $|H| = nm$ for some integer m. Let $p_1 ... p_r$ be a prime factorization of m. Then p_1 divides $|H|$, so by hypothesis H has a subgroup H_1 of index p_1 in H. Noting that $|H_1| = np_2...p_r$, p_2 divides $|H_1|$, so by hypothesis, H_1 contains a subgroup H_2 of index p_2. Continuing this way, we see that for $i = 2, ..., r$, there is a subgroup H_i of index p_i in H_{i-1}. Furthermore,

$$|H_r| = |H|/(H : H_r) = |H|/((H : H_1)(H_1 : H_2)...(H_{r-1} : H_r))$$

$$= nm/(p_1 p_2 ... p_r) = nm/m = n.$$

Thus H_r is the desired subgroup. □

Theorem 4.2 *A group G is supersoluble if and only if every subgroup of G satisfies* clt.

By 4.1 it suffices to prove:

Theorem 4.3 *A group G is supersoluble if and only if for every subgroup H of G, H has a subgroup of index p for every prime p dividing $|H|$.*

Proof: $\Rightarrow$ We shall use induction on $|G|$, G a supersoluble group. If $H < G$ then H is supersoluble and by induction contains a subgroup of prime index q in H, for every prime q dividing $|H|$. It therefore remains to show that if q is a prime divisor of $|G|$ then G possesses a subgroup of index q.

Let p be the largest prime dividing $|G|$. By 3.2(a) a Sylow p-subgroup S of G is normal in G and there is a complement T of S in G.

If

$$1 = G_0 \leq G_1 \leq ... \leq G_a = G$$

is any supersoluble series of G, set $S_i = G_i \cap S$ and take a supersoluble series

$$S/S = S_a/S \leq S_{a+1}/S \leq ... \leq S_{a+b}/S = G/S$$

of G/S. Then

$$1 = S_0 \leq S_1 \leq ... \leq S_{a+b} = G$$

is a supersoluble series of G containing S as a term. Since we can refine this to a supersoluble series of G with factors of prime order (as in the proof of 1.5(a)) and S is a p-group, we can choose $P < G$ such that $P \leq S$ and $(S : P) = p$.

We have $q \le p$. If $q < p$ then consider the quotient G/S. G/S is supersoluble and $|G/S| < |G|$. Thus by induction there is a subgroup K/S of G/S of index q. But then K is a subgroup of G with $(G : K) = (G/S : K/S) = q$. In this case, K is the required subgroup.

Assume therefore that $q = p$. Set $M = PT$. Now $P \cap T \le S \cap T = 1$. So

$$|M| = |PT| = |P||T|/|P \cap T| = |P||T|.$$

And $|G| = |ST| = |S||T|$. Thus we have

$$(G : M) = |S||T|/|P||T| = |S|/|P| = (S : P) = p, = q.$$

So in this situation, M is the required subgroup.

$\Rightarrow$ We again use induction. Let q be the smallest prime dividing $|G|$. By assumption there is a subgroup K of G with $(G : K) = q$. Since q is the smallest prime dividing G we have $K \lhd G$, by 0.1(b). By induction, K is supersoluble.

We may assume that K is non-trivial; otherwise G has order q, is cyclic and so supersoluble. Let p be the largest prime dividing $|K|$. Using 3.2, let S be the normal Sylow p-subgroup of K. Since it is unique, it is characteristic in K and thus normal in G.

We have $p \ge q$. If $p = q$ then G must be a p-group; for q is the smallest prime dividing $|G|$, so the only prime dividing $|K|$ is $p = q$ and $(G : K) = q$. Thus G is supersoluble.

If $p > q$ then as p does not divide $(G : K)$, S is a Sylow p-subgroup of G. $\zeta_1 S$ is a non-trivial normal subgroup of G (S is a non-trivial p-group and $\zeta_1 S$ is characteristic in S). Thus we can choose a minimal subgroup N of G which lies inside $\zeta_1 S$.

We claim that

(a) $|N| = p$. In particular, we claim that N is a cyclic normal subgroup;

(b) G/N is supersoluble.

By hypothesis, G contains a subgroup M of index p. M must be a maximal subgroup of G. By the maximality of M, $MN = M$ or $MN = G$.

Suppose $MN = G$. Since N is abelian, we have $M \cap N \lhd N$. As $N \lhd G$ we have $M \cap N \lhd M$. Thus $M, N \le N_G(M \cap N)$. Hence $M \cap N \lhd MN = G$. Also $M \cap N \le N$, so the minimality of N gives $M \cap N = N$ or 1. If $M \cap N = N$ then

$$|G| = |MN| = |M||N|/|M \cap N| = |M|,$$

contradiction. So $M \cap N = 1$. Thus we have

$$|N| = |MN|/|M| = (G : M) = p.$$

If $MN = M$ then $N \le M$. M is supersoluble by induction, so we know that N must contain a subgroup N_1 of order p which is normal in M. Then $M \le N_G(N_1)$ and since $N_1 \le N \le \zeta_1 S$, S centralizes N_1, so in particular, S normalizes N_1. S is not contained in M, so the maximality of M gives $SM = G$.

Thus $G = N_G(N_1)$, or in other words we have $N_1 \leq G$. Using the minimality of N, $N = N_1$. Thus $|N| = p$. We have therefore proved claim (a), in either case.

Let $H/N < G/N$. By induction, H is supersoluble and so H/N is supersoluble. By the necessity argument above, H/N contains a subgroup of prime index r for each prime divisor r of $|H/N|$.

If r is a prime dividing $|G/N|$ then r divides $|G|$. Thus G contains a subgroup R such that $(G : R) = r$. If $N \leq R$ then R/N is a subgroup of G/N with

$$(G/N : R/N) = (G : R) = r.$$

In this case, by induction G/N is supersoluble, establishing (b).

We now consider the case where N is not contained in R. Since N is cyclic of prime order and N is not contained in R we have $N \cap R = 1$. $(G : R)$ is prime, so R is a maximal subgroup of G. $R < RN$, so that $G = RN$. Then

$$G/N = RN/N \cong R/R \cap N \cong R, < G.$$

R is supersoluble by induction and so G/N is supersoluble, again giving (b).

(a) and (b) yield that G is cyclic-by-supersoluble and so supersoluble, by 1.2. □

In chapter 1 we showed that a (not necessarily finite) supersoluble group had maximal subgroups (1.7) and that they each have prime index (1.9). The latter property provides a characterization of finite supersoluble groups and this was shown by Huppert. There are several proofs of this result. Some of these use results from representation theory which we wish to avoid. For alternative proofs to the on we give here, see either [4] 10.5.8 or [10] 9.4.4. We prove some auxilary results.

Proposition 4.4 *If G is strictly p-closed for some prime p, then G is supersoluble.*

Before proving 4.4, we note that G does not have to be strictly p-closed for every prime p, to be supersoluble. For example, Sym(3) is supersoluble and is strictly 3-closed but not strictly 2-closed (it does not have a normal Sylow 2-subgroup).

Proof: We proceed by induction on $|G|$. Let S be a Sylow p-subgroup of G, p being some prime for which G is strictly p-closed. If $S = 1$ then $G \cong G/S$ is abelian (of exponent dividing $p - 1$) and thus is supersoluble. So consider the case where $S \neq 1$.

Set $Z = \zeta_1 S$. Since S is a p-group, we have $Z \neq 1$. Also Z is normal in G. Thus Z contains a minimal normal subgroup N of G. $S \leq C_G(N)$, since $N \leq Z$.

N is an elementary abelian p-group by 0.10(a). Since G/S is abelian of exponent dividing $p-1$, $G/C_G(N) \cong (G/S)/(C_G(N)/S)$ is abelian of exponent dividing $p - 1$. By 0.5(b), N is cyclic of order p. Then G/N has order less than that of G. S/N has order p^{r-1}, where $p^r = |S|$. So S/N is the Sylow p-subgroup of G/N ($S/N \lhd G/N$ since $S \lhd G$), and $(G/N)/(S/N) \cong G/S$ is abelian of exponent dividing $p-1$. Thus G/N is strictly p-closed. By induction, G/N is supersoluble. Thus G is cyclic-by-supersoluble and therefore supersoluble, by 1.2. □

Before proving Huppert's Theorem, we note an interesting corollary of 4.4:

Corollary 4.5 *If G has order qp^r where p and q are primes with q dividing $p-1$, then G is supersoluble. In particular, a group of order $2p^r$ is supersoluble.*

Proof: Using Sylow's Theorem, if S is a Sylow p-subgroup of G then it must be unique, since the number of Sylow q-subgroups is congruent to $1 \bmod p$, must divide q, which divides $p-1$. Hence $S \lhd G$. The order of G/S is q, so that G/S is abelian (it is cyclic) of exponent dividing $p-1$. By 4.4, G is supersoluble. □

Theorem 4.6 (Huppert c1954) *If the maximal subgroups of G all have prime index then G is supersoluble.*

Proof: We first show that G is soluble[1]. Choose p to be the largest prime divisor of $|G|$. Let S be a Sylow p-subgroup of G. S is nilpotent and so soluble. Suppose that S is not normal in G. Then $N_G(S)$ is contained in a maximal subgroup M of G. $(G : N_G(S))$ is coprime to p (it divides $(G : S)$) and $(G : M)$ divides $(G : N_G(S))$. Thus $(G : M)$ is coprime to p, or in other words, $(G : M) = 1 \bmod p$. But $(G : M) = q$ is prime. By choice of p, we must have $p > q, > 1$. But then $(G : M) \neq 1 \bmod p$. This is a contradiction. So $S \lhd G$.

Now M/S is a maximal subgroup of G/S if and only if M is a maximal subgroup of G and M contains S. Moreover, $(G/S : M/S) = (G : M)$ is prime. So G/S satisfies the hypothesis. Since $|G/S| < |G|$, G/S is soluble by induction. Thus G is soluble-by-soluble and hence soluble.

We now show that G is supersoluble. Choose a minimal normal subgroup H of G. In a similar way to above, G/H satisfies the hypothesis of the Theorem and $|G/H| < |G|$, so that by induction G/H is supersoluble. If K is a minimal normal subgroup of G that is different from H, then also G/K is supersoluble by induction. Further $H \cap K \lhd G$ and $H \cap K \lhd K$. Thus, by the minimality of K, we have $H \cap K = 1$. Then $G \cong G/H \cap K$ which is supersoluble by 1.4(b). Therefore we may assume that H is the unique minimal normal subgroup of G.

The solubility of G ensures that H is an elementary abelian p-group by 0.10(a). Then H is a normal nilpotent subgroup of G and so it is contained in the Fitting subgroup $\eta_1 G$. If q is a prime dividing $|\eta_1 G|$ and $q \neq p$, then let Q be a Sylow q-subgroup of $\eta_1 G$. The nilpotency of $\eta_1 G$ yields that Q is the unique Sylow q-subgroup of $\eta_1 G$. Thus Q is characteristic in $\eta_1 G$ and hence normal in G. But then G has a normal q-subgroup. Thus is must have a minimal normal subgroup that is a q-group. This contradicts the uniqueness of H. Thus we assume that $\eta_1 G$ is a p-group.

If H is not contained in the Frattini subgroup ΦG, then there is a maximal subgroup M of G such that H is not contained in M. The maximality of M

---

[1] In this step we prove a special case of a theorem of Philip Hall, namely: If all the maximal subgroups of finite G have prime or square of a prime index, then G is soluble.

yields $HM = G$, since $M \lhd HM$. $M \cap H \lhd H$ since H is abelian, and $H \lhd G$ so that $M \cap H \lhd M$. Thus H and M normalize $M \cap H$ and hence $M \cap H \lhd HM = G$. But $M \cap H \lhd H$. Thus, by the minimality of H, $M \cap H = 1$. Then

$$(G : M) = (HM : M) = (H : H \cap M) = (H : 1) = |H|.$$

Thus H must be cyclic. But then H is cyclic-by-supersoluble and hence supersoluble by 1.2. So we assume that $H \leq \Phi G$.

Since H is non-trivial, so is ΦG. It follows that by induction $G/\Phi G$ is supersoluble. By 0.9(c) $\eta_1(G/\Phi G) = \eta_1 G/\Phi G$. Thus $\eta_1(G/\Phi G)$ is a p-group. By 1.10(a), $(G/\Phi G)/\eta_1(G/\Phi G)$ is abelian. Since $\eta_1(G/\Phi G)$ is a p-group, any chief factor of $G/\Phi G$ of order coprime to p is centralized by $G/\Phi G$. This fact, together with 0.9(d), yields that $\eta_1(G/\Phi G)$ is the intersection of the centralizers in $G/\Phi G$ of chief factors of $G/\Phi G$ whose order is p. If C is one of these centralizers, by 0.3 and 0.5(a) $(G/\Phi G)/C$ is abelian of exponent dividing $p-1$. It follows that

$$G/\eta_1 G \cong (G/\Phi G)/(\eta_1 G/\Phi G) = (G/\Phi G)/\eta_1(G/\Phi G)$$

is abelian of exponent dividing $p - 1$.

Since $\eta_1 G$ is a p-group, it is contained in a Sylow p-subgroup S of G. Thus $G' \leq \eta_1 G \leq S$. Therefore $S \lhd G$. And

$$G/S \cong (G/\eta_1 G)/(S/\eta_1 G).$$

Thus G/S is abelian of exponent dividing $p - 1$. Hence G is strictly p-closed and so by 4.4, G is supersoluble. □

Corollary 4.7 *(a) G is supersoluble if and only if every maximal subgroup of G has prime index.*

(b) If $N \lhd G$ and N is contained in every maximal subgroup of G then G is supersoluble if and only if G/N is supersoluble.

(c) If $L \lhd G$ then G is supersoluble if and only if $G/\Phi L$ is supersoluble. In particular, G is supersoluble if and only if $G/\Phi G$ is supersoluble.

Proof: (a) This is the union of results 4.6 and 1.9.

(b) Suppose that G/N is supersoluble. Let M be a maximal subgroup of G. By hypothesis, N is contained in M. M/N is a maximal subgroup of G/N. By (a), $(G : M) = (G/N : M/N)$ is prime. Thus by (a) again G is supersoluble.

(c) If $L \lhd G$ then $\Phi L \leq \Phi G$ by 0.9(b), and ΦG is contained in every maximal subgroup of G, by definition. ΦL is characteristic in L and thus is normal in G. (c) then follows from (b) by taking $N = \Phi L$. □

The next result due to Kramer involves maximal subgroups and the Fitting subgroup.

Theorem 4.8 (Kramer c1976) *Let G be soluble. Then G is supersoluble if and only if for every maximal subgroup M of G, either $\eta_1 G \leq M$ or $M \cap \eta_1 G$ is a maximal subgroup of $\eta_1 G$.*

Note that the solubility of G is required in Kramer's Theorem. Alt(5) is an insoluble simple group. Since the Fitting subgroup is a normal nilpotent subgroup, we must have $\eta_1(\text{Alt}(5)) = 1$. Thus $\eta_1(\text{Alt}(5))$ is contained in every subgroup of Alt(5) and so in particular, in every maximal subgroup. It is clear also that Alt(5) is not supersoluble.

Proof: $\Rightarrow$ Assume that G is supersoluble. By 1.9, if M is a maximal subgroup of G then $(G : M)$ is prime. If the Fitting subgroup $\eta_1 G$ is not contained in M, then since $M < M\eta_1 G$, the maximality of M ensures that $G = M\eta_1 G$. But then

$$(\eta_1 G : \eta_1 G \cap M) = (M\eta_1 G : M) = (G : M).$$

Thus $(\eta_1 G : \eta_1 G \cap M)$ is prime so that $\eta_1 G \cap M$ must be a maximal subgroup of $\eta_1 G$.

$\Leftarrow$ If $M/\Phi G$ is a maximal subgroup of $G/\Phi G$ then M is a maximal subgroup of G. Thus $M \geq \eta_1 G$ or $M \cap \eta_1 G$ is a maximal subgroup of $\eta_1 G$. But by 0.9(c), $\eta_1 G/\Phi G = \eta_1(G/\Phi G)$. Thus $M/\Phi G \geq \eta_1(G/\Phi G)$ or

$$(M \cap \eta_1 G)/\Phi G = (M/\Phi G) \cap (\eta_1 G/\Phi G) = (M/\Phi G) \cap \eta_1(G/\Phi G)$$

is a maximal subgroup of $\eta_1(G/\Phi G)$. Thus the hypothesis is also satisfied by $G/\Phi G$. Hence if $\Phi G \neq 1$, by induction $G/\Phi G$ is supersoluble and then by 4.7(c) G is supersoluble. Thus assume that $\Phi G = 1$.

By 0.10(b) $\eta_1 G$ is abelian and is the direct product of (abelian) minimal normal subgroups of G, say

$$\eta_1 G = H_1 \times H_2 \times ... \times H_r.$$

As $\Phi G = 1$, for each $i = 1, 2, ..., r$ there is a maximal subgroup M_i of G such that H_i is not contained in M_i. $M_i < M_i H_i$, so that the maximality of M_i gives us that $M_i H_i = G$. Since H_i is abelian and since $H_i \lhd G$, it follows that $M_i \cap H_i \lhd M_i H_i = G$. The minimality of H_i yields that $M_i \cap H_i = 1$, for each $i = 1, ..., r$.

Also for each $i = 1, ..., r$, we have

$$\eta_1 G = G \cap \eta_1 G = H_i M_i \cap \eta_1 G = H_i(M_i \cap \eta_1 G),$$

using the Modular law. If $\eta_1 G \leq M_i$ then $H_i \leq M_i$, contradiction. Thus by hypothesis $M_i \cap \eta_1 G$ must be a maximal subgroup of M. By 1.9, $(\eta_1 G : M_i \cap \eta_1 G)$ is prime. Then we have

$$|H_i| = (H_i : M_i \cap H_i) = (H_i M_i : M_i) = (G : M_i)$$

$$= ((\eta_1 G)M_i : M_i) = (\eta_1 G : M_i \cap \eta_1 G).$$

Thus $|H_i|$ is prime. Hence by 0.3 and 0.5, $G/C_G(H_i)$ is abelian. Therefore, we have $G' \leq C_G(H_i)$. Thus

$$G' \leq \bigcap_{i=1,\dots,r} C_G(Hi) = C_G(\eta_1 G), \leq \eta_1 G$$

by 0.10(c).

Now let M be a maximal subgroup of G. Then either $\eta_1 G \leq M$ or $M\eta_1 G = G$. If $\eta_1 G \leq M$, then $G' \leq M$. Thus $M \lhd G$ and G/M is an abelian simple group. Hence $(G : M)$ is prime. If $M\eta_1 G = G$ then

$$(G : M) = (M\eta_1 G : M) = (\eta_1 G : M \cap \eta_1 G)$$

which is prime by 1.9. Thus in all cases, $(G : M)$ is prime. By Huppert's Theorem (4.6), G is supersoluble. □

Corollary 4.9 *Let G be soluble. Then G is supersoluble if and only if for every maximal subgroup M of G and each $N \lhd G$, either M contains N or $M \cap N$ is a maximal subgroup of N.*

Proof: Let M be a maximal subgroup of supersoluble group G. If N is not contained in M then the maximality of M yields $MN = G$. Therefore $(G : M) = (MN : M) = (N : M \cap N)$ is prime by 4.7(a). Thus $M \cap N$ is a maximal subgroup of N.

For the converse, suppose that for every maximal subgroup M of G and every $N \lhd G$, either M contains N or $M \cap N$ is a maximal subgroup of N. In particular, this must hold for the normal subgroup $\eta_1 G$. By Kramer's Theorem (4.8), G is supersoluble. □

Definition. A *maximal subgroup chain* is a sequence of subgroups[2] of G

$$1 = G_0 < G_1 < ... < G_n = G$$

where G_{i-1} is a maximal subgroup of G_i for $i = 1, ..., n$. Equivalently, a maximal subgroup chain of G is a sequence of subgroups with no proper refinements. Note that a composition series of a soluble group is an example of a maximal subgroup chain.

G is called *equichained* if all maximal subgroup chains of G have the same length.

Let

$$1 = H_0 < H_1 < ... < H_n = H$$

and

$$1 = J_0 < J_1 < ... < J_m = H$$

be maximal subgroup chains of H, a subgroup of an equichained group G, then since G is finite, we can complete these chains to maximal subgroup chains of G, say:

$$1 = H_0 < H_1 < ... < H_n = H < L_1 < ... < L_s = G$$

and

$$1 = J_0 < J_1 < ... < J_m = H < L_1 < ... < L_s = G.$$

[2] WARNING: We do not require this sequence to be a *series* of G; i.e. we do not require each $G_{i-1} \lhd G_i$.

Since G is equichained, we have $n + s = m + s$ and hence $n = m$. Thus H is equichained. We have therefore shown that a subgroup of an equichained group is equichained.

We now aim to show that an equichained group is supersoluble and conversely. This was discovered by Iwasawa. The proof requires some auxilary results.

Definition. If p is a prime, G is p-normal if whenever S and T are Sylow p-subgroups with $\zeta_1 S \leq T$ then $\zeta_1 S = \zeta_1 T$.

Lemma 4.10 *If G is not p-normal then the centre of a Sylow p-subgroup is always nonnormal in some other Sylow p-subgroup.*

Proof: By negating the definition of p-normal, there exist Sylow p-subgroups S, T such that $\zeta_1 S \neq \zeta_1 T$, but $\zeta_1 S \leq T$. Suppose for a contradiction that $\zeta_1 S \lhd T$. Then both S and T normalize $\zeta_1 S$ and then both S and T are Sylow p-subgroups of $N_G(\zeta_1 S)$. By Sylow's Theorem, there is $g \in N_G(\zeta_1 S)$ with $S^g = T$. But then $\zeta_1 S = (\zeta_1 S)^g = \zeta_1(S^g) = \zeta_1 T$. This is a contradiction. Thus $\zeta_1 S$ cannot be normal in T. □

We state but do not prove the following two results - their proofs would be out of context here.

Theorem 4.11 (Grün) *Let G be a p-normal group and S be a Sylow p-subgroup of G. The largest abelian p-group which occurs as a factor group of G is isomorphic to the largest abelian p-group which occurs as a factor group of $N_G(\zeta_1 S)$.*

Proof: This can be found in [13] 13.5.4. □

Theorem 4.12 (Burnside) *Let p be a prime, H a p-subgroup of G such that H is normal in some Sylow p-subgroup of G but is nonnormal in some other Sylow p-subgroup of G. Then there is a p-subgroup L of G such that $N_G(L)/C_G(L)$ is not a p-group.*

Proof: This is a reformulation part of Theorem IV.2.u in [12]. □

The next result, which is essential in the development we have chosen to follow of Iwasawa's Theorem, is of independent interest.

Theorem 4.13 (Huppert c1954) *Suppose G is a group whose proper subgroups are supersoluble. Then G is soluble.*

Proof: We induct on the order of G, noting that the result is vacuous for trivial groups. So assume that G is non-trivial and every proper subgroup of G is supersoluble. Every proper subgroup of every proper factor group of G is supersoluble so that by induction every proper factor group is soluble. Thus if we can show that G is not simple, G will be the extension of a supersoluble group by a soluble group and thus will be soluble.

Suppose p is the smallest prime dividing $|G|$. If G is a p-group then G is nilpotent and hence soluble, so suppose that G is not a p-group.

Let L be a non-trivial p-subgroup of G. If $L \lhd G$ then G is not simple and we have finished, so suppose that $N_G(L) < G$. Then $N_G(L)$ is supersoluble. By 3.1 $N_G(L)$ has a Sylow tower. Using 3.2(b), let R be a normal complement in $N_G(L)$ to any Sylow p-subgroup of $N_G(L)$. R, L are both normal subgroups of $N_G(L)$, so $[R, L] \leq R \cap L$. $R \cap L = 1$ by Lagrange's Theorem, because p does not divide $|R|$ and L is a p-group. Thus $R \leq C_G(L)$. $N_G(L)/R$ is a p-group and hence

$$N_G(L)/C_G(L) \cong (N_G(L)/R)/(C_G(L)/R)$$

is a p-group. By 4.12 there can be no p-subgroup of H of G that is normal in one Sylow p-subgroup but is nonnormal in some other. By 4.10, G must be p-normal.

Let P be a Sylow p-subgroup of G. In a similar way to above we can assume that $N_G(\zeta_1 P)$ is a proper subgroup G and hence is supersoluble. Note that since $P \leq N_G(\zeta_1 P)$, P is a Sylow p-subgroup of $N_G(\zeta_1 P)$. Again we can use 3.2(b) to get, $N_G(\zeta_1 P) = XP$ where X is a normal complement to P in $N_G(\zeta_1 P)$.

Considering the subgroup XP' of $N_G(\zeta_1 P)$, for any $x \in X$ we have

$$(P')^x = x^{-1}P'x \subset XP'X = XXP' = XP'.$$

Thus for $x \in X$ and $p \in P$,

$$(XP')^{xp} = X^{xp}P'^{xp} = X(P'^{xp}) = X(XP')^p = XX^pP'^p = XXP' = XP',$$

since $X < N_G(\zeta_1 P) = XP$ and $P' < P$. So $XP' < N_G(\zeta_1 P)$.

Note that $XP' \cap P = P'$. For $z \in XP' \cap P$ implies that $z = xp$ for some $x \in X$ and $p \in P'$. Then $x = zp^{-1} \in PP' \subset P$. But $x \in X$ and $X \cap P = 1$, whence $zp^{-1} = 1$ and then $z = p \in P$. Conversely, $P' \leq XP'$ and $P' \leq P$, so $P' \leq XP' \cap P$.

Now

$$N_G(\zeta_1 P)/XP' = XP/XP' = XP'P/XP' \cong P/XP' \cap P = P/P',$$

is a non-trivial abelian p-group. By 4.11, G has a non-trivial abelian quotient. Thus G cannot be simple, which completes the proof. □

Theorem 4.14 (Iwasawa c1941) *The following are equivalent:*

(a) G is supersoluble.

(b) G is equichained.

(c) The length of each maximal subgroup chain of G is equal to the number of prime divisors of $|G|$.

Proof: (c) $\Rightarrow$ (b): is clear. (b) $\Rightarrow$ (a): Suppose that G is non-trivial; for the result is vacuous if $G = 1$. If $H < G$ then H is equichained. By induction, H is supersoluble for any $H < G$. By 4.13, G is soluble. Thus G has a composition series with cyclic factors of prime order. Since G is soluble, composition series of G is a maximal subgroup chain of G. It follows that any maximal subgroup must have prime order. By 4.6, G must be supersoluble.

(a) $\Rightarrow$ (c): Again, this is vacuous for $G = 1$. So suppose G is non-trivial. G is supersoluble, so any maximal subgroup M has prime index in G by 1.9 (or 4.7). Since M is supersoluble, the length of each maximal subgroup chain of M is equal to the number of prime divisors of $|M|$. Thus a maximal subgroup chain in which M occurs as a term has length equal to the number of prime divisors of $|G|$. The result follows as M was any maximal subgroup of G. □

Historical Note O. Ore in [9] showed that G is a group whose subgroups and quotients satify *clt* if and only if G is soluble and has conformal chains - that is, any two maximal subgroup chains have the same length and the magnitudes of the factors are the same in possibly a different order. Ore conjectured that it was enough for the subgroups of G to satisfy *clt* for G to have conformal chains and it was G. Zappa who proved this(see [17]).

Since a composition series of a soluble group has cyclic of prime order factors and is a maximal subgroup chain, the condition on the magnitudes of the factors becomes redundant. Thus soluble groups with conformal chains are precisely the equichained groups. Iwasawa was the first to realize that the equichained groups are precisely the supersoluble groups. One could use these facts to obtain 4.2.

Chapter 5

Further Results

In this chapter, we revert to the situation where G is not necessarily finite. We present some miscellaneous results regarding supersoluble groups. We shall not give full proofs to some of these results - we shall either direct the reader to a reference or merely indicate a proof.

Other finiteness conditions

Supersoluble groups satisfy other finiteness conditions other than *max*. These generally follow from the fact that a supersoluble group is polycyclic-by-finite.

G is called *finitely presented* if it has a presentation consisting of finitely many generators and relations.

A cyclic group has a presentation with one generator and at most one relation. Thus the cyclic groups are finitely presented. It is a theorem of Philip Hall (see [10] 2.2.4) that a finitely presented-by-finitely presented group is finitely presented. Thus using induction on the length of a supersoluble series, one can obtain:

5.1 *A supersoluble group is finitely presented.* □

G is *residually finite* if the following equivalent conditions hold:

1) For every $1 \neq g \in G$, there is $N_g \lhd G$ such that $g \notin N_g$ and G/N_g is finite.

2) $\bigcap\{N : N \lhd G \quad \text{and} \quad G/N \quad \text{is finite}\} = 1.$

5.2 *A supersoluble group is residually finite.*

Proof: We induct on the Hirsch number of a group G, $h(G)$. If $h(G) = 0$ then G is obviously finite and the result holds (take $N_g = 1$ for every $g \in G$ in the definition above). Thus assume that $h(G) > 0$.

By 2.8(b), G has an infinite free abelian normal subgroup, A, say. For any natural number m, $A^m \lhd G$ and $(A : A^m)$ is finite. Thus we have $h(G/A^m) = h(G) - h(A^m)$. It is clear that $h(A^m) = h(A) > 0$. Thus $h(G/A^m) < h(G)$.

By induction,

$$\bigcap\{N/A^m : N/A^m \lhd G/A^m \quad \text{and} \quad G/N \cong (G/A^m)/(N/A^m) \quad \text{is finite}\} = A^m/A$$

That is,

$$\bigcap\{N : A^m \leq N \lhd G \quad \text{and} \quad G/N \quad \text{is finite}\} \leq A^m.$$

Taking the intersection over all natural numbers m we get

$$\bigcap\{N : N \lhd G \quad \text{and} \quad G/N \quad \text{is finite}\} \leq 1.$$

□

One can show that polycyclic-by-finite groups are finitely presented and are residually finite.

Images

Given a polycyclic group G, the amount of supersolubility of its finite homomorphic images control the amount of supersolubility of G, in the following sense:

Theorem 5.3 (Baer) *If G is polycyclic and every finite homomorphic image of G is supersoluble, then G is supersoluble.*

Proof: see [15] 11.11. □

Hypercyclic groups

A *system* of G is a sequence of subgroups of G, $(G_\alpha)_{0\leq\alpha\leq\beta}$, where β is some ordinal, such that

$$1 = G_0 \leq G_1 \leq ... \leq G_\beta = G,$$

$G_\alpha \lhd G$, for all $\alpha \leq \beta$ and if λ is a limit ordinal, then $G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$. As with series, we call the G_α *terms*, the $G_{\alpha+1}/G_\alpha$ *factors*, etc. Note that a finite system is a series.

A *hypercyclic system* is a system with cyclic factors. We call G *hypercyclic* if it possesses a hypercyclic system. Clearly, a supersoluble group is hypercyclic. For a hypercyclic group to be supersoluble, it must at least be finitely generated. In fact, this is enough to guarantee supersolubility.

Theorem 5.4 (Baer) *A hypercyclic group is supersoluble if and only if it is finitely generated.*

Proof: For a proof of this see either [15] 11.10 or [12] VII.7.g.□

More on *clt* groups

In chapter 4, we showed that a finite group G is supersoluble if and only if every subgroup of G is *clt*. In [7], J. F. Humphreys proved a similar result regarding the factor groups of G.

Theorem 5.5 (Humphreys) *If G is a finite group of odd order all of whose factors are* clt*, then G is supersoluble.* □

One cannot drop the hypothesis that G has odd order. For example, Sym(4) is a group of even order with every factor group *clt*, but it is not supersoluble.

In [8], McLain showed that the supersolubility of finite group G is in some sense controlled by the existence of subgroups between characteristic subgroups of G.

Theorem 5.6 (McLain) *Let G be a finite group. G is supersoluble if and only if between any two characteristic subgroups $H > K$, there exist subgroups of every possible order.* □

Generalized Central Series

Given a finite group G, $g \in G$ is called a *generalized central* element of G if $< g > P = P < g >$ (or equivalently $< g > P \leq G$) for every Sylow subgroup P of G.

Set

$$\Xi G =< g \in G : g \quad \text{is a generalized central element of} \quad G >,$$

and call ΞG, the *generalized centre* of G. One can easily show that ΞG is a normal subgroup of G. One can also show that ΞG is nilpotent.

The method used to define the upper central series, can be used to define the *upper generalized central series* of G as follows: Let $\xi_0 G = 1$, and then for $i \geq 0$, let $\xi_{i+1}G$ be the subgroup of G such that $\xi_{i+1}G/\xi_i G = \Xi(G/\xi_i G)$.

The *hyper generalized centre* of G is then $\xi G = \bigcup_i \xi_i G$, which since G is finite must equal some term $\xi_m G$.

Theorem 5.7 (Agrawal [1]) *Let G be a finite group. The following are equivalent:*

(a) G is supersoluble.

(b) $\xi_n G = G$ for some n.

(c) $\xi G = G$.

Proof: (b) ⇒ (c): This is obvious. (a) ⇒ (b): G has a normal non-trivial cyclic subgroup $< x >$, say, by 1.5(b). The normality of $< x >$ ensures that

$< x > P = P < x >$ for every Sylow subgroup P of G. Thus x is a non-identity element that is a generalized central element. Therefore $\Xi G \neq 1$, for any supersoluble group G.

It follows that

$$1 = \xi_0 G < \xi_1 G < \xi_2 G < \xi_3 G < \ldots.$$

If $\xi_n G$ is the hyper generalized center of G and $\xi_n G \neq G$, then $\xi_n G < \xi_{n+1} G$, since $G/\xi_n G$ is supersoluble and $\Xi(G/\xi_n G)$ is non-trivial, contradicting the fact that $\xi_n G$ was the last term in the series. So $\xi_n G = G$.

(c) $\Rightarrow$ (a): This is the hardest part of the proof and we shall only give an outline. For a complete proof see [1] 2.8.

The result is true for $G = 1$, so assume that G is non-trivial.

Several facts hold when $G = \xi G$, namely:

(i) $\xi(G/K) = G/K$ for every $K \lhd G$.

(ii) G has a Sylow tower.

(iii) ΞG is non-trivial.

By (i), and using induction, every proper quotient of G is supersoluble. Thus, if the Frattini subgroup ΦG is non-trivial, G is supersoluble using 4.7(c). So we may as well assume that $\Phi G = 1$. Using (ii), G has a normal Sylow p-subgroup P for p the largest prime dividing the order of G. Also using a simple induction argument, one can show that G is soluble because it has a Sylow tower. The fact that $\Phi P \leq \Phi G = 1$ is enough to ensure that P is abelian.

We now aim to use Huppert's Theorem, 4.6, to complete the proof. Let M be a maximal subgroup in G. The solubility of G gives that $(G : M)$ is a power of a prime.

If $(G : M)$ is not a power of p, then M contains a Sylow p-subgroup and so P is contained in M. But then $(G : M) = (G/P : M/P)$. Since G/P is supersoluble, by induction, $(G : M)$ is prime and so G is supersoluble.

Suppose that $(G : M)$ is a power of p. Let q be another prime that divides the order of ΞG. If Q is a Sylow q-subgroup of ΞG, then $Q < \Xi G$. This is because ΞG is nilpotent. Thus Q is characteristic in ΞG and so normal in G. It follows that $Q \leq M$ and then $(G : M) = (G/Q : G/M)$. Thus $(G : M)$ is prime and G is supersoluble. Thus we may assume that ΞG is a p-group.

Since ΞG is generated by generalized central elements and powers of generalized central elements are generalized central elements, G must have a generalized central element of order p. Set

$$N = < g : g \quad \text{a generalized central element of } G \text{ of order } p > .$$

$N \lhd G$. If $N \leq M$, then since $(G : M) = (G/N : M/N)$, G is supersoluble.

If N is not contained in M then there is a generalized central element y of order p that is not contained in M. $< y >$ is a p-group, so $< y > \leq P$. Since P is abelian, $< y > \lhd P$.

One can show that the elements of G whose orders are p'- numbers, also normalize $< y >$. Thus $< y > \lhd G$, and since $M < M < y >$, we have

$G = M < y >$. Hence $(G : M)$ is the order of y, which is p. Thus G is supersoluble by Huppert's Theorem. □

FINIS

Bibliography

[1] R. K. Agrawal, Generalized center and hypercenter of a finite group, *Proc. AMS* **58**, 1976, pp13-21.

[2] T. S. Blyth & E. F. Robertson, *Essential Student Algebra Volume 5: Groups*, Chapman & Hall, 1986.

[3] W. E. Deskins, A characterization of finite supersoluble groups, *Amer. Math. Monthly* **75**, 1968, pp180-2.

[4] M. Hall, *The Theory of Groups*, The Macmillan Company, 1959.

[5] P. Hall, Theorems like Sylow's, *Proc. LMS* **6**, 1956, pp286-304.

[6] C. V. Holmes, A characterization of finite nilpotent groups, *Amer. Math. Monthly* **73**, 1966, pp1113-4.

[7] J. F. Humphreys, On groups satifying the converse of Langrange's Theorem, *Proc. Camb. Phil. Soc.* **75**, 1974, pp25-32.

[8] D. H. McLain, The existence of subgroups of given order in finite groups, *Proc. Camb. Phil. Soc.* **53**, 1957, pp278-85.

[9] O. Ore, Contributions to the theory of finite groups, *Duke Math. J.* **5**, 1939, pp431-60.

[10] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, 1982.

[11] J. S. Rose, *A Course on Group Theory*, Dover Publications, 1994.

[12] E. Schenkman, *Group Theory*, Van Nostrand, 1965.

[13] W. R. Scott, *Group Theory*, Dover Publications, 1987.

[14] D. Segal, *Polycyclic Groups*, Cambridge University Press, 1983.

[15] B. A. F. Wehrfritz, *Infinite Linear Groups*, Springer-Verlag, 1973.

[16] M. Weinstein (editor), *Between Nilpotent and Solvable*, Polygonal Publishing House, 1982.

[17] G. Zappa, A remark on a recent paper of O. Ore, *Duke Math. J.* **6**, 1940, pp511-2.